

# STATIC SOLUTIONS TO THE EINSTEIN-VLASOV SYSTEM WITH NON-VANISHING COSMOLOGICAL CONSTANT

HÅKAN ANDRÉASSON, DAVID FAJMAN, MAXIMILIAN THALLER

**ABSTRACT.** We construct spherically symmetric, static solutions to the Einstein-Vlasov system with non-vanishing cosmological constant  $\Lambda$ . The results are divided as follows. For small  $\Lambda > 0$  we show existence of globally regular solutions which coincide with the Schwarzschild-deSitter solution in the exterior of the matter sources. For  $\Lambda < 0$  we show via an energy estimate the existence of globally regular solutions which coincide with the Schwarzschild-Anti-deSitter solution in the exterior vacuum region. We also construct solutions with a Schwarzschild singularity at the center regardless of the sign of  $\Lambda$ . For all solutions considered, the energy density and the pressure components have bounded support. Finally, we point out a straightforward method to obtain a large class of globally non-vacuum spacetimes with topologies  $\mathbb{R} \times S^3$  and  $\mathbb{R} \times S^2 \times \mathbb{R}$  which arise from our solutions using the periodicity of the Schwarzschild-deSitter solution. A subclass of these solutions contains black holes of different masses.

## 1. INTRODUCTION

Schwarzschild's construction of a static explicit solution in 1915 was the first example of a solution to Einstein's field equations in general relativity [27]. It has been found later that the class of static, spherically symmetric, asymptotically flat solutions to Einstein's equations in vacuum consists only of that element [13] making it necessary to consider the non-vacuum field equations to construct further classes of spherically symmetric static spacetimes.

**1.1. Static solutions with Vlasov matter.** In this work we consider matter described as a collisionless gas. In astrophysics this model is used to study galaxies and globular clusters where the stars, or the galaxies, are the particles of the gas and where collisions between these are sufficiently rare to be neglected. The particles interact by the gravitational field which the particle ensemble creates collectively. Within the framework of general relativity the particle system is described by the Einstein-Vlasov system. The mathematical investigation of this system was initiated by Rein and Rendall in 1992 [25] in the context of the Cauchy problem and shortly thereafter the same authors provided the first study of static, spherically symmetric solutions to this system [24]. Since then, the Einstein-Vlasov system has been successfully studied in several contexts and many global results have been obtained during the last two decades. We refer to [2] for a review of these results but let us in particular mention the recent monumental work on this system concerning the stability of the universe [26].

The purpose of the present work is to extend the class of static solutions to the Einstein-Vlasov system to the case with a non-vanishing cosmological constant  $\Lambda$ . Several results on static and stationary solutions to this system have been obtained in the case when  $\Lambda = 0$ . The first result of this kind was provided in [24], where the authors construct spherically symmetric isotropic static solutions with compactly supported energy density and pressure. The solutions are asymptotically flat and thus serve as models for isolated, self-gravitating systems. Several generalizations of this result have since then been obtained, in particular solutions with non-isotropic pressure, and solutions with a

---

*Date:* Friday 19<sup>th</sup> September, 2014.

*1991 Mathematics Subject Classification.* 83C05, 83C20, 83C57.

*Key words and phrases.* Einstein equations, Einstein-Vlasov system, static solutions, Schwarzschild-deSitter, Schwarzschild-Anti-deSitter, Black holes.

Schwarzschild singularity at the center, have been established, cf. [23, 21]. An approach by variational methods was developed by Wolansky [28]. The most difficult part in these proofs is to show that the matter has compact support. A neat and quite general method to treat this problem has recently been obtained by Ramming and Rein in [20]. However, this method does not straightforwardly apply to the situation we consider in this work. The cosmological constant changes the structure of the equations and this implies that inequality (1.23) in [20], on which this method is based, does not hold when  $\Lambda \neq 0$ . Hence, we rely on a different method in this work. The results discussed above all concern the spherically symmetric case. Let us point out that results beyond spherical symmetry have been established. The existence of stationary axially symmetric solutions to the Einstein-Vlasov system has recently been shown, cf. [9] and [10] for the non-rotating and the rotating case respectively. In this context we also mention a result on static solutions for elastic matter which has been obtained without any symmetry assumption [1].

**1.2. Static solutions with non-vanishing cosmological constant.** A specific class of solutions has so far not been discussed which concerns the Einstein equations with a non-vanishing cosmological constant  $\Lambda$ . The model solutions for the vacuum equations are the Schwarzschild-deSitter and Schwarzschild-Anti-deSitter (Schwarzschild-AdS) solution for  $\Lambda > 0$  and  $\Lambda < 0$ , respectively. Einstein's equations with non-vanishing  $\Lambda$  are of significant physical interest, where the case  $\Lambda > 0$  applies to a universe with accelerated expansion [26], while the case  $\Lambda < 0$  is relevant in the context of AdS-CFT correspondence [18]. Concerning the Einstein-Vlasov system no existence results for the static Einstein equations with non-vanishing cosmological constant are known. The aim of the present paper is to prove existence of spherically symmetric static solutions to the Einstein-Vlasov system with small positive or arbitrary negative cosmological constant. The solutions we construct are in general anisotropic. The results provided in this work are as follows.

**1.2.1. Globally regular solutions for  $0 < \Lambda \ll 1$ .** We construct globally regular static solutions for small  $\Lambda > 0$ . The fundamental difference to the case of vanishing cosmological constant is that for large radii the metric tends towards a cosmological horizon and it is thus necessary to show that the support of the matter quantities vanishes before the cosmological horizon is reached. We show that for small  $\Lambda > 0$  the solutions we construct are close to the solutions corresponding to the  $\Lambda = 0$  case for which the matter quantities have compact support, and in addition, the latter solutions obey a Buchdahl type inequality. These facts imply that the support of the matter quantities can be controlled also in the case when  $\Lambda > 0$ . It is then possible to continue the solution from the vacuum region by a Schwarzschild-deSitter solution. This method yields a large class of globally regular solutions which coincide with a Schwarzschild-deSitter solution outside a compact set. The result is given in Theorem 3.7.

**1.2.2. Globally regular solutions for  $\Lambda < 0$ .** The case of negative cosmological constant is a priori simpler since the cosmological term has a good sign which yields a monotonically decreasing behavior of the lapse function. An energy argument following the general idea of [23] is used to establish global in  $r$  existence yielding globally regular solutions for general  $\Lambda < 0$ . The result is given in Theorem 4.2.

**1.2.3. Solutions with a Schwarzschild singularity for  $0 < \Lambda \ll 1$ .** To construct solutions with singularities in the center, we start with the vacuum equations which can be solved explicitly by the Schwarzschild-deSitter solution. This solution is considered until a radius which allows to continue the vacuum solution by one which at the same point satisfies the ansatz for the distribution function and eventually merges into a non-vacuum region. It is shown that the support of the matter quantities is compact and outside the matter region the solution can again be extended by a vacuum solution with mass parameter corresponding to the interior mass of black hole and matter. As in the non-singular case

these constructions only work out for sufficiently small  $\Lambda > 0$ . The result is given in Theorem 5.5.

1.2.4. *Solutions with a Schwarzschild singularity for  $\Lambda < 0$ .* This point is similar to the case  $\Lambda > 0$  with Schwarzschild singularities. In particular a smallness condition for  $|\Lambda|$  is needed as well. The result is given in Theorem 5.9.

1.2.5. *Solutions with topologies  $\mathbb{R} \times S^3$  and  $\mathbb{R} \times S^2 \times \mathbb{R}$ .* A significant generalization of the results with  $\Lambda > 0$  is presented in the final section. The periodic structure of the Schwarzschild-deSitter space [17] allows us to consider solutions with regular massive center, and solutions with central black holes, and glue them to a periodic Schwarzschild-deSitter solution with a black hole region followed by another matter region - forming a static space-time with two non-vacuum ends and a black hole (or several) in between. The result is given in Theorem 6.1.

1.3. **Outline of the paper.** This paper is organized as follows. In Section 2 we introduce the notation and give a short review on the static Einstein-Vlasov system in spherical symmetry. We discuss the anisotropic ansatz for the distribution function, variations of which are used in this work. A Buchdahl type inequality, which applies to solutions of the Einstein-Vlasov system, is then reviewed shortly as it is used later in the existence proof for  $\Lambda > 0$ . The Einstein-Vlasov system in spherical symmetry with a specific ansatz for the distribution function reduces to an integro-differential equation given in (2.24). This equation lies at the heart of the analysis in the paper. In Section 3 we prove existence of globally regular solutions for small  $\Lambda > 0$ . The proof is divided into several steps beginning with local in  $r$  existence in 3.1, a continuation criterion in 3.2, existence for sufficiently large radii to reach a vacuum region in 3.3 and finally the proof of the existence Theorem in 3.4. In Section 4 the existence of globally regular solutions for arbitrary  $\Lambda < 0$  is proven along with a result (cf. Theorem 4.2) which states existence for such solutions outside a ball, which eventually is used to prove existence of solutions with Schwarzschild singularities in the center. Section 5 begins with a generalization of the Buchdahl type inequality, mentioned above, for solutions with Schwarzschild singularities. This result is useful for the construction of solutions of this kind when  $\Lambda > 0$ . These solutions are obtained in Theorem 5.5. Analogous solutions for the case of negative  $\Lambda$  are given in Theorem 5.9. Finally, Section 6 discusses the globally non-trivial generalizations of the constructed solutions for  $\Lambda > 0$ .

### Acknowledgements

D.F. and M.T. are grateful to Walter Simon and Bobby Beig for several discussions on static solutions. We thank Piotr Chrusciel for the suggestion to study the case of negative cosmological constants. We are indebted to Greg Galloway for sharing his ideas and suggestions concerning the global solutions which we present in the last chapter of this work. We thank Christa Olz for helpful discussions. D.F. thanks Amos Ori and Gershon Wolansky for interesting discussions.

## 2. PRELIMINARIES

2.1. **Setup and notations.** We consider the Einstein-Vlasov system with cosmological constant  $\Lambda \in \mathbb{R}$ . For background on this system we refer to [2]. For the spherically symmetric, static Lorentzian metric  $g$  we use the standard ansatz

$$(2.1) \quad ds^2 = -e^{2\mu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2(\vartheta) d\varphi^2.$$

Assuming in addition the matter distribution function  $f$  to be spherically symmetric and static one obtains the reduced system of equations

$$(2.2) \quad \frac{v^a}{\sqrt{1+|v|^2}} \frac{\partial f}{\partial x^a} - \sqrt{1+|v|^2} \mu' \frac{x^a}{r} \frac{\partial f}{\partial v^a} = 0,$$

$$(2.3) \quad e^{-2\lambda}(2r\lambda' - 1) + 1 - r^2\Lambda = \kappa r^2 \varrho,$$

$$(2.4) \quad e^{-2\lambda}(2r\mu' + 1) - 1 + r^2\Lambda = \kappa r^2 p,$$

where  $\kappa = 8\pi$ ,  $|v| = \sqrt{\delta_{ij}v^i v^j}$ ,  $v_r = \frac{\delta_{ij}v^i x^j}{r}$  and the matter quantities read

$$(2.5) \quad \varrho = \int_{\mathbb{R}^3} f(x, v) \sqrt{1+|v|^2} dv^1 dv^2 dv^3,$$

$$(2.6) \quad p = \int_{\mathbb{R}^3} \frac{f(x, v)}{\sqrt{1+|v|^2}} v_r^2 dv^1 dv^2 dv^3.$$

There is an additional Einstein equation

$$(2.7) \quad e^{-2\lambda} \left( \mu'' \left( \mu + \frac{1}{r} \right) (\mu' - \lambda') \right) = \kappa p_T,$$

where

$$(2.8) \quad p_T = \frac{1}{2} \int_{\mathbb{R}^3} \left| \frac{x \times v}{r} \right|^2 f(x, v) \frac{dv}{\sqrt{1+|v|^2}}.$$

The quantity  $\varrho$  can be understood as energy density,  $p$  as radial pressure and  $p_T$  as tangential pressure. To ensure a regular center the following boundary condition is imposed

$$(2.9) \quad \lambda(0) = 0.$$

This condition will be used in the first part of this work but when we consider solutions with a Schwarzschild singularity at the center it will be dropped. A detailed derivation of the system (2.2)-(2.8) in the  $\Lambda = 0$  case can be found in [25]. It will be seen below that a solution of the reduced system (2.2)-(2.6) also solves the full system. Considering the characteristic curves of the Vlasov equation (2.2) one can simplify the system of equations. Along these characteristic curves the quantities  $E$  and  $L$ , given by

$$(2.10) \quad E = e^{\mu(r)} \sqrt{1+|v|^2} =: e^{\mu(r)} \varepsilon \quad \text{and} \quad L = |x \times v|^2,$$

are conserved (cf. [24]). Therefore any ansatz for the matter distribution  $f$  of the form

$$(2.11) \quad f(x, v) = \Phi(E, L)$$

solves the Vlasov equation (2.2), and this equation drops out from the system of equations.

**2.2. Relevant results.** In the following we discuss the known results for the Einstein-Vlasov system with vanishing cosmological constant,  $\Lambda = 0$ , which are relevant for the work presented in this paper. The existence of a unique solution  $\mu(r)$ ,  $\lambda(r)$  to given initial values  $\mu(0) = \mu_0$  and  $\lambda(0) = 0$  has been proved using the ansatz

$$(2.12) \quad f(x, v) = \Phi(E)[L - L_0]_+^\ell,$$

where  $E > 0$ ,  $L > 0$ ,  $L_0 \geq 0$ ,  $\ell > -\frac{1}{2}$ ,  $\Phi \in L^\infty((0, \infty))$  for the matter distribution  $f$ , cf. [23]. Furthermore, it can be shown that the support of the matter quantities is contained in an interval  $[0, R_0]$ ,  $0 < R_0 < \infty$ , if one takes a so called *polytropic* ansatz for  $f$ . This ansatz has the form

$$(2.13) \quad f(x, v) = \phi \left( 1 - \frac{E}{E_0} \right) L^\ell,$$

where  $\phi : \mathbb{R} \rightarrow [0, \infty)$  is measurable,  $\phi(\eta) = 0$  for  $\eta < 0$ , and  $\phi > 0$  a.e. on some interval  $[0, \eta_1]$  with  $\eta_1 > 0$  and  $E_0$  is some prescribed cut-off energy, cf. [20]. Moreover, it is

required that there exists  $\gamma > -1$  such that for every compact set  $K \subset \mathbb{R}$  there exists a constant  $C > 0$  such that

$$(2.14) \quad \phi(\eta) \leq C\eta^\gamma, \quad \eta \in K.$$

In [22] this result is generalized to anisotropic matter distributions of the form

$$(2.15) \quad f(x, v) = c_0[E_0 - E]_+^k [L - L_0]_+^\ell,$$

where  $k \geq 0$ ,  $\ell > -1/2$  fulfill the inequality  $k < 3\ell + 7/2$  and  $c_0, E_0 > 0$ ,  $L_0 \geq 0$ . It is shown in [22] that for sufficiently small  $L_0$  the support of  $f$  is contained in an interval  $[R_i, R_0]$  where  $0 \leq R_i < R_0 < \infty$  and  $R_i > 0$  provided  $L_0 > 0$ .

By direct calculation one shows that the matter quantities fulfill the generalized Tolman-Oppenheimer-Volkov equation (TOV equation)

$$(2.16) \quad p'(r) = -\mu'(r)(p(r) + \varrho(r)) - \frac{2}{r}(p(r) - p_T(r)).$$

Another result relevant for the proof presented here is a generalized Buchdahl inequality [4], which is the content of the following lemma.

**Lemma 2.1** (Theorem 1 in [4]). *Let  $\lambda, \mu \in C^1([0, \infty))$  and let  $\varrho, p, p_T \in C^0([0, \infty))$  be functions that satisfy the system of equations (2.3)-(2.7), the condition (2.9) and such that  $p + 2p_T \leq \varrho$ . Then*

$$(2.17) \quad \sup_{r>0} \frac{2m(r)}{r} \leq \frac{8}{9},$$

where

$$(2.18) \quad m(r) = 4\pi \int_0^r s^2 \varrho(s) ds.$$

**Remark 2.2.** *The inequality (2.17) holds for a more general class of functions, cf. [4]. Moreover, the inequality is sharp, and the solutions which saturate the inequality are infinitely thin shell solutions, cf. [4]. In [3] it is shown that there exist regular, arbitrarily thin, shell solutions to the Einstein-Vlasov system such that the quantity  $2m/r$  can be arbitrarily close to  $8/9$ . It should also be mentioned that Buchdahl type inequalities have been obtained in the case of non-vanishing cosmological constant, cf. [6, 7]. These results assume the existence of static solutions to the Einstein-matter equations with a cosmological constant.*

To prove existence of solutions of the static Einstein-Vlasov system with non-vanishing  $\Lambda$  we make use of the results discussed above. To simplify calculations we define  $y := \ln(E_0) - \mu$  as in [20] so that  $e^\mu = E_0/e^y$ . For the distribution function  $f$  we choose the ansatz<sup>1</sup>

$$(2.19) \quad \begin{aligned} f(x, v) &= \Phi(E, L) = c_0 \phi \left( 1 - \frac{E}{E_0} \right) [L - L_0]_+^\ell \\ &= c_0 \phi (1 - \varepsilon e^{-y}) [L - L_0]_+^\ell, \\ \phi(\eta) &= [\eta]_+^k, \end{aligned}$$

where  $k \geq 0$ ,  $\ell > -1/2$  fulfill the inequality  $k < 3\ell + 7/2$  and  $c_0, E_0 > 0$ ,  $L_0 \geq 0$ . For the construction of globally regular solutions  $L_0$  has to be sufficiently small to ensure finite support of the matter quantities [22]. When considering solutions with a black hole at the center, there are positive lower bounds on  $L_0$ . The expressions for the matter quantities  $\varrho$  and  $p$  take the form

$$(2.20) \quad \varrho(r) = G_\phi(r, y(r)), \quad p(r) = H_\phi(r, y(r)),$$

<sup>1</sup>To be precise any  $\phi$  that is of the kind of the  $\phi$  in (2.13) would meet the assumptions of the following lemmas and theorems.

where

$$(2.21) \quad G_\phi(r, y) = c_\ell c_0 r^{2\ell} \int_{\sqrt{1+L_0/r^2}}^{\infty} \phi(1 - \varepsilon e^{-y}) \varepsilon^2 \left( \varepsilon^2 - \left( 1 + \frac{L_0}{r^2} \right) \right)^{\ell + \frac{1}{2}} d\varepsilon,$$

$$(2.22) \quad H_\phi(r, y) = \frac{c_\ell c_0}{2\ell + 3} r^{2\ell} \int_{\sqrt{1+L_0/r^2}}^{\infty} \phi(1 - \varepsilon e^{-y}) \left( \varepsilon^2 - \left( 1 + \frac{L_0}{r^2} \right) \right)^{\ell + \frac{3}{2}} d\varepsilon,$$

given in [23]. The constant  $c_\ell$  is given by

$$(2.23) \quad c_\ell = 2\pi \int_0^1 \frac{s^\ell}{\sqrt{1-s}} ds.$$

**Lemma 2.3.** *The functions  $G_\phi(r, y)$  and  $H_\phi(r, y)$  defined in (2.21) and (2.22), respectively, have the following properties.*

- (i)  $G_\phi(r, y)$  and  $H_\phi(r, y)$  are continuously differentiable in  $r$  and  $y$ .
- (ii) The functions  $G_\phi(r, y)$  and  $H_\phi(r, y)$  and the partial derivatives  $\partial_y G_\phi(r, y)$  and  $\partial_y H_\phi(r, y)$  are increasing both in  $r$  and  $y$ .
- (iii) There is vacuum, i.e.  $f(r, \cdot) = p(r) = \varrho(r) = 0$  if  $e^{-y(r)} \sqrt{1 + L_0/r^2} \geq 1$ , in particular if  $y(r) \leq 0$ .

*Proof.* By performing a change of variables in the integrals in (2.21) and (2.22) the differentiability follows, cf. [23], Lemma 3.1. The monotonicity can be seen directly from the structure of  $G_\phi$  and  $H_\phi$ . The last statement is obvious since  $\phi(\eta) = 0$  if  $\eta \leq 0$ .  $\square$

**2.3. Main equation.** From the Einstein equations (2.3) and (2.4) one obtains the differential equation for  $y$

$$(2.24) \quad y'(r) = - \frac{\kappa/2}{1 - \frac{\Lambda r^2}{3} - \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, y(s)) ds} \times \left( r H_\phi(r, y(r)) - \frac{2r\Lambda}{3\kappa} + \frac{1}{r^2} \int_0^r s^2 G_\phi(s, y(s)) ds \right).$$

A solution to (2.24) yields a solution to the system (2.2-2.6). It should however be pointed out that in order to obtain an asymptotically flat solution one needs to redefine  $E_0$  and  $\mu$  as follows. Given an initial value  $y_0$ , a solution  $y$  of equation (2.24) is obtained having a limit  $y(\infty)$ . By letting  $E_0 := 1/y(\infty)$  and  $e^\mu := E_0 y(r)$  we get a solution with the proper boundary condition at infinity. Furthermore it should be mentioned that a solution to the system (2.2-2.6) provides a solution to all the Einstein equations. This is shown in Theorem 2.1 in [25] in the case when  $\Lambda = 0$ . The proof is analogous in the case with non-vanishing  $\Lambda$ . The equation (2.24) is analyzed and solved in the remainder of this work.

### 3. STATIC, ANISOTROPIC GLOBALLY REGULAR SOLUTIONS FOR $\Lambda > 0$

In this section we prove existence of globally regular static solutions with small  $\Lambda > 0$ .

**3.1. Local existence.** The following local existence lemma corresponds to the first part of the proof of Theorem 2.2 in [24] for the case  $\Lambda = 0$ .

**Lemma 3.1.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) and let  $G_\phi, H_\phi$  be defined by equations (2.21) and (2.22), respectively. Then for every  $y_0 \in \mathbb{R}$  and every  $\Lambda > 0$  there is a  $\delta > 0$  such that there exists a unique solution  $y_\Lambda \in C^2([0, \delta])$  of equation (2.24) with initial value  $y_\Lambda(0) = y_0$ .*

*Proof.* We consider the equation (2.24) and integrate it using the initial condition  $y_\Lambda(0) = y_0$ . The following fixed point problem is obtained,

$$(3.1) \quad y_\Lambda(r) = (Ty_\Lambda)(r), \quad r \geq 0$$

where the operator  $T$  is given by

$$(3.2) \quad (Tu)(r) := y_0 - \int_0^r \frac{\kappa/2}{1 - \frac{s^2\Lambda}{3} - \frac{\kappa}{s} \int_0^s \sigma^2 G_\phi(\sigma, u(\sigma)) d\sigma} \times \left( sH_\phi(s, u(s)) - \frac{2s\Lambda}{3\kappa} + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, u(\sigma)) d\sigma \right) ds.$$

This operator is considered on the set

$$(3.3) \quad M := \left\{ u : [0, \delta] \rightarrow \mathbb{R} \mid u(0) = y_0, y_0 - 1 \leq u(r) \leq y_0 + 1, \right. \\ \left. \frac{r^2\Lambda}{3} + \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, u(s)) ds \leq c < 1, r \in [0, \delta] \right\}.$$

We note that  $M$  is non-empty if  $\delta > 0$  is chosen sufficiently small. As carried out in detail in the appendix, Section A, it is shown that  $T$  acts as a contraction on  $M$ . This implies (by the Banach fixed-point theorem) that there exists  $y_\Lambda \in M$  such that  $Ty_\Lambda = y_\Lambda$ . Differentiability of  $y_\Lambda$  follows from the structure of  $T$ . The differentiation with respect to  $r$  yields that  $y_\Lambda$  solves equation (2.24) on the interval  $[0, \delta]$ . Away from the singularity  $r = 0$ , standard existence and uniqueness results are applied to extend  $y_\Lambda$  to a maximal solution on an interval  $[0, R_c)$ . Obviously, the boundary condition at  $r = 0$  is satisfied. The regularity of the functions  $G_\phi$  and  $H_\phi$  implies that  $y_\Lambda \in C^2((0, R_c))$ , (cf. [23]) and it can be shown that the second derivative continuously extends to  $r = 0$  and  $y'_\Lambda(0) = 0$ .  $\square$

**3.2. Continuation criterion.** The solution  $y_\Lambda$  exists at least as long as the denominator of the right hand side of equation (2.24) is strictly larger than zero. The following lemma formulates this assertion.

**Lemma 3.2.** *Let  $y_0 \in \mathbb{R}$  and let  $R_c > 0$  be the largest radius such that the unique local  $C^2$ -solution  $y_\Lambda$  of equation (2.24) with  $y_\Lambda(0) = y_0$  exists on the interval  $[0, R_c)$ . Then there exists  $R_D \leq R_c$  such that*

$$(3.4) \quad \liminf_{r \rightarrow R_D} \left( 1 - \frac{r^2\Lambda}{3} - \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds \right) = 0.$$

**Remark 3.3.** *Lemma 3.2 implies that the denominator on the right hand side of equation (2.24) becomes arbitrarily small on  $[0, R_c)$ , i.e. the numerator has no singular behavior that would make the solution collapse as long as the denominator is larger than zero.*

**Remark 3.4.** *We can a priori not exclude the case  $R_c = \infty$  which would however not occur due to the  $\Lambda$  term.*

*Proof.* Assume

$$(3.5) \quad 1 - \frac{r^2\Lambda}{3} - \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds > 0$$

for all  $r \in [0, R_c)$ . Otherwise  $R_D < R_c$  (with  $R_D$  characterized as above) occurs due to the continuity of  $y_\Lambda$  and  $G_\phi$  and the lemma follows. Assume now that the assertion of the lemma does not hold, i.e. there is a constant  $a > 0$  such that

$$(3.6) \quad 1 - \frac{r^2\Lambda}{3} - \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds \geq a$$

for all  $r \in [0, R_c)$ . First we show that this implies the existence of a  $C > 0$  such that for all  $r \in [0, R_c)$  we have  $|y'_\Lambda(r)| \leq C$ . Therefore we consider

$$(3.7) \quad |y'_\Lambda(r)| \leq \frac{4\pi}{a} \left( rH_\phi(r, y_\Lambda(r)) + \frac{2r\Lambda}{3\kappa} + \frac{1}{r^2} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds \right).$$

Here it is used that  $H_\phi$  and  $G_\phi$  are positive. It is obvious that the second term,  $\frac{2r\Lambda}{3\kappa}$ , is bounded on the interval  $[0, R_c)$ . We show that the right hand side of (3.7) is bounded on this interval. Assume the opposite,

$$(3.8) \quad \limsup_{r \rightarrow R_c} H_\phi(r, y_\Lambda(r)) = \infty \quad \text{or} \quad \limsup_{r \rightarrow R_c} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds = \infty.$$

The second possibility implies  $\limsup_{r \rightarrow R_c} G_\phi(r, y_\Lambda(r)) = \infty$ . On the interval  $[0, R_c)$  we have the upper bounds  $H_\phi(r, y_\Lambda(r)) \leq H_\phi(R_c, y_\Lambda(r))$  and  $G_\phi(r, y_\Lambda(r)) \leq G_\phi(R_c, y_\Lambda(r))$ , cf. Lemma 2.3, (ii). And since  $H_\phi(r, y)$  and  $G_\phi(r, y)$  are increasing functions in  $y$  (cf. Lemma 2.3) this in turn implies

$$(3.9) \quad \limsup_{r \rightarrow R_c} y_\Lambda(r) = \infty.$$

It follows that for all  $\varepsilon > 0$  sufficiently small there exists  $r \in (R_c - \varepsilon, R_c)$  such that  $y'_\Lambda(r) > 0$  which on the other hand implies

$$(3.10) \quad rH_\phi(r, y_\Lambda(r)) + \frac{1}{r^2} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds < \frac{2r\Lambda}{3\kappa},$$

by equation (2.24) for  $y'_\Lambda$ . This contradicts the assumption that either  $H_\phi(r, y_\Lambda(r))$  or the integral  $\int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds$  diverge as the right hand side of (3.10) is bounded. Thus  $|y'_\Lambda(r)|$  is bounded on  $[0, R_c)$ .

In the remainder of this proof it is shown that the solution can be continued beyond  $R_c$  which yields the desired contradiction. To achieve this, similar methods as in the proof of Lemma 3.1 will be used. For  $\delta, \varepsilon > 0$ ,  $\delta > \varepsilon$  define  $y_\varepsilon = y_\Lambda(R_c - \varepsilon)$ , the interval  $I_\varepsilon$  containing  $R_c$  by  $I_\varepsilon = [R_c - \varepsilon, R_c - \varepsilon + \delta]$ , and

$$(3.11) \quad u_y(r) := \begin{cases} y_\Lambda(r); & r \in [0, R_c - \varepsilon] \\ u(r); & r > R_c - \varepsilon \end{cases}.$$

Consider the operator

$$(3.12) \quad (T_\varepsilon u)(r) = y_\varepsilon + \int_{R_c - \varepsilon}^r \frac{\kappa/2}{1 - \frac{s^2\Lambda}{3} - \frac{\kappa}{s} \int_0^s \sigma^2 G_\phi(\sigma, u_y(\sigma)) d\sigma} \times \left( sH_\phi(s, u(s)) - \frac{2s\Lambda}{3\kappa} + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, u_y(\sigma)) d\sigma \right) ds$$

acting on the set

$$(3.13) \quad M_\varepsilon = \left\{ u : I_\varepsilon \rightarrow \mathbb{R} \mid u(R_c - \varepsilon) = y_\varepsilon, y_\varepsilon - 1 \leq u(r) \leq y_\varepsilon + 1, \right. \\ \left. \frac{r^2\Lambda}{3} + \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, u_y(s)) ds \leq c < 1, r \in I_\varepsilon \right\}.$$

Using (3.6) and  $|y'_\Lambda(r)| < C$  on  $[0, R_c)$  for a  $C > 0$  one can prove that  $T_\varepsilon$  acts as a contraction on  $M_\varepsilon$ . In virtue of Banach's fixed point theorem the operator  $T_\varepsilon$  has a fixed point  $y_\varepsilon \in M_\varepsilon$  such that  $(y_\varepsilon)_y$  defined by (3.11) solves equation (2.24) on the interval  $(0, R_c - \varepsilon + \delta)$ . But this contradicts the definition of  $R_c$  and the lemma follows.  $\square$



### 3.3. Existence beyond the non-vacuum region.

**Lemma 3.5.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) and let  $y$  be the unique global  $C^1$ -solution of equation (2.24) in the case  $\Lambda = 0$  where  $y(0) = y_0 > 0$ , cf. [23]. As proved in [23],  $f$  has bounded spatial support  $[0, R_0]$  where  $y(R_0) = 0$  defines  $R_0$  uniquely. Let  $y_\Lambda$  be the unique  $C^2$ -solution of equation (2.24) with  $\Lambda > 0$  and  $y_\Lambda(0) = y(0)$  that according to Lemma 3.1 exists at least on an interval  $[0, \delta]$  for a certain  $\delta > 0$  and let  $f_\Lambda$  be the distribution function corresponding to  $y_\Lambda$ .*

*Then  $y_\Lambda$  exists at least on  $[0, R_0 + \Delta R]$  and the spatial support of  $f_\Lambda$  is bounded by some  $R_{0,\Lambda} < R_0 + \Delta R$  if  $\Lambda$  and  $\Delta R > 0$  are chosen such that*

$$(3.14) \quad 0 < \Lambda < \min \left\{ \frac{|y(R_0 + \Delta R)|}{C_y(R_0 + \Delta R)}, \frac{\frac{1}{18}}{C_v(R_0 + \Delta R)} \right\}$$

*holds. The constants  $C_y(r)$  defined in equation (3.24) and  $C_v(r)$  defined in equation (3.22) are determined by the background solution  $y$ .*

**Remark 3.6.** *Note that the upper bound for  $\Lambda$  in (3.14) is strictly larger than zero since  $|y(R_0 + \Delta R)| > 0$ . This holds because the globally existing background solution  $y$  is strictly monotone and we have  $y(R_0) = 0$  by definition of  $R_0$ .*

*Proof.* We define

$$(3.15) \quad m(r) = 4\pi \int_0^r s^2 \varrho(s) ds, \quad m_\Lambda(r) = 4\pi \int_0^r s^2 \varrho_\Lambda(s) ds,$$

$$(3.16) \quad v(r) = 1 - \frac{2m(r)}{r}, \quad v_\Lambda(r) = 1 - \frac{r^2 \Lambda}{3} - \frac{2m_\Lambda(r)}{r}.$$

Consider the continuous function  $v_\Lambda$ . Note that  $v_\Lambda(0) = 1$ . We define

$$(3.17) \quad r^* := \inf \{r \in [0, R_c] \mid v_\Lambda(r) = 1/18\},$$

i.e.,  $r^*$  is the smallest radius where  $v_\Lambda(r) = \frac{1}{18}$ . Lemma 3.2 assures that  $r^* < R_c$ , i.e.,  $r^*$  is well defined. Note that  $v_\Lambda(r)$  is the quantity in Lemma 3.2. In addition, we define

$$(3.18) \quad \tilde{r} := \sup \{r \in [0, R_c] \mid |y_\Lambda(r) - y(r)| \leq |y(R_0 + \Delta R)|\}.$$

The right hand side of this inequality is given by the background solution  $y$ , which exists globally. Note that  $|y(R_0 + \Delta R)| > 0$  since  $y$  is strictly monotone, and  $y(0) = y_\Lambda(0) = y_0$ , so  $0 < \tilde{r}$  by continuity of  $y$  and  $y_\Lambda$ . Let

$$(3.19) \quad \tilde{r}^* := \min\{r^*, \tilde{r}\}.$$

Choosing  $\Lambda$  s.t. (3.14) holds, we will show that  $\tilde{r}^* > R_0 + \Delta R$ . We assume the opposite,  $\tilde{r}^* \leq R_0 + \Delta R$ , and consider the sum  $|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|$  on the interval  $[0, \tilde{r}^*]$ . By the mean value theorem we have

$$(3.20) \quad |\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| = \left( \left| \partial_y G_\phi(r, y) \right|_{u_1} + \left| \partial_y H_\phi(r, y) \right|_{u_2} \right) |y_\Lambda(r) - y(r)|$$

where  $u_1, u_2 \in [y(r), y_\Lambda(r)]$  are chosen appropriately. From the estimate (B.2) in Appendix B we have that for  $r \leq \tilde{r}^*$

$$(3.21) \quad |\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| \leq \Lambda C_{gh}(\tilde{r}^*),$$

where  $C_{gh}$  is defined in (B.2). Note that  $C_{gh}(r)$  is increasing in  $r$ . Still on  $[0, \tilde{r}^*]$  we compute

$$(3.22) \quad \begin{aligned} |v(r) - v_\Lambda(r)| &\leq \frac{r^2 \Lambda}{3} + \frac{2}{r} |m_\Lambda(r) - m(r)| = \frac{r^2 \Lambda}{3} + \frac{8\pi}{r} \int_0^r s^2 |\varrho_\Lambda(s) - \varrho(s)| ds \\ &\leq \left( \frac{(\tilde{r}^*)^2}{3} + \frac{8\pi}{3} (\tilde{r}^*)^2 C_{gh}(\tilde{r}^*) \right) \Lambda =: C_v(\tilde{r}^*) \Lambda \end{aligned}$$

Since we have  $v(r) \geq \frac{1}{9}$  (Buchdahl inequality, cf. Lemma 2.1) and  $\Lambda < \frac{1/18}{C_v(R_0 + \Delta R)}$  by choice of  $\Lambda$  we can conclude

$$(3.23) \quad v_\Lambda(r) \geq v(r) - \Lambda C_v(\tilde{r}^*) > \frac{1}{9} - \frac{1/18}{C_v(R_0 + \Delta R)} C_v(\tilde{r}^*) \geq \frac{1}{18}$$

on  $[0, \tilde{r}^*]$  since  $C_v(\tilde{r}^*) < C_v(R_0 + \Delta R)$  because  $C_v(r)$  is increasing and  $\tilde{r}^* \leq R_0 + \Delta R$  by assumption.

We also consider the distance between  $y$  and  $y_\Lambda$  on  $[0, \tilde{r}^*]$ . Following the procedure depicted in Section B of the appendix one obtains

$$(3.24) \quad \begin{aligned} |y_\Lambda(r) - y(r)| &\leq \Lambda \left( 3r^2 + 29\pi r^4 \left( H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \\ &\quad + 72\pi \left( r + 24\pi r^2 \left( H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \int_0^r C_{gh}(s) \Lambda ds \\ &=: C_y(r) \Lambda \leq C_y(\tilde{r}^*) \Lambda. \end{aligned}$$

Since  $C_y(\tilde{r}^*) \leq C_y(R_0 + \Delta R)$  and  $\Lambda < \frac{|y(R_0 + \Delta R)|}{C_y(R_0 + \Delta R)}$  on  $[0, \tilde{r}^*]$  by assumption, the relation

$$(3.25) \quad |y_\Lambda(r) - y(r)| < |y(R_0 + \Delta R)|$$

already holds. Equations (3.23) and (3.25) state that  $v_\Lambda(\tilde{r}^*) > \frac{1}{18}$  and  $|y_\Lambda(\tilde{r}^*) - y(\tilde{r}^*)| < |y(R_0 + \Delta R)|$ , respectively on the interval  $[0, \tilde{r}^*]$ , which is a contradiction to the definition of  $\tilde{r}^*$ . Thus we have  $\tilde{r}^* > R_0 + \Delta R$  as desired.

We have shown that  $y_\Lambda$  exists at least on  $[0, R_0 + \Delta R]$  as the continuation criterion applies and from equation (3.25) we already know that  $y_\Lambda(R_0 + \Delta R) < 0$ . Since  $y_\Lambda$  is continuous it has at least one zero at in the interval  $(R_0, R_0 + \Delta R)$ . In particular there exists an interval  $(R_{0\Lambda}, R_0 + \Delta R)$  where  $y_\Lambda$  is strictly smaller than zero.  $R_{0\Lambda}$  is the largest zero of  $y_\Lambda$  in  $(R_0, R_0 + \Delta R)$ . So the spatial support of  $f_\Lambda$  is contained in the interval  $[0, R_{0\Lambda})$  and this implies the assertion.  $\square$

**3.4. Global regular solutions for  $\Lambda > 0$ .** In the last two sections we have seen that for suitably chosen  $\Lambda$  there exists a unique solution  $y_\Lambda$  to equation (2.24) on the interval  $[0, R_0 + \Delta R]$  for some  $\Delta R > 0$ . This solution uniquely induces a solution  $\mu_\Lambda, \lambda_\Lambda$  of the equations (2.3), (2.4) on  $[0, R_0 + \Delta R]$  whose distribution function  $f_\Lambda$  is of bounded support in space. By gluing a Schwarzschild-deSitter metric to this solution one can construct a global static solution to the Einstein-Vlasov system.

**Theorem 3.7.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19). For every initial value  $\mu_0 < 0$  there exists a constant  $C = C(\mu_0, \phi) > 0$  such that for every  $0 < \Lambda < C$  there exists a unique global solution  $\mu_\Lambda, \lambda_\Lambda \in C^2([0, \infty))$ ,  $f_\Lambda \in C^0([0, \infty) \times \mathbb{R}^3)$  of the static, spherically symmetric Einstein-Vlasov system (2.2)-(2.6) with  $\mu_\Lambda(0) = \mu_0$ , and  $\lambda_\Lambda(0) = 0$  such that the support of the distribution function is bounded. This solution coincides with the Schwarzschild-deSitter metric in the vacuum region.*

*Proof.* According to Lemma 3.1 there exists a  $C^2$ -solution  $y_\Lambda$  of equation (2.24) on a small interval  $[0, \delta]$ . In the proof of Lemma 3.5 we saw that this solution can be extended at least until  $r = R_0 + \Delta R$  for any  $\Delta R$  if one chooses  $\Lambda$  small enough. Beyond the support of  $\varrho_\Lambda$  and  $p_\Lambda$ , thus for  $r \in [R_{0,\Lambda}, R_0 + \Delta R]$ , equation (2.24) takes the form

$$(3.26) \quad y'_\Lambda(r) = -\frac{1}{2} \frac{d}{dr} \ln \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M}{r} \right)$$

where  $M = m_\Lambda(R_{0,\Lambda})$ . This equation is solved by the (shifted) Schwarzschild-deSitter metric, whose corresponding  $y$ -coefficient  $y_S$  is given by

$$(3.27) \quad y_S(r) = -\frac{1}{2} \ln \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M}{r} \right) - \ln \left( e^{-\lambda(R_{0,\Lambda})} \right).$$

The shift has been chosen such that  $y_\Lambda$  can be extended by  $y_S$  as a  $C^2$ -solution of equation (2.24) on  $[0, \infty)$  using a modified ansatz for the matter distribution  $f_\Lambda$ . Namely, for  $r > R_0 + \Delta R$  we drop the original ansatz  $\Phi$  for  $f_\Lambda$  and continue  $f_\Lambda$  by the constant zero function, i.e.

$$(3.28) \quad f_\Lambda(x, v) = \begin{cases} [1 - \varepsilon e^{-y}]_+^k [L - L_0]_+^\ell, & r \in [0, R_0 + \Delta R] \\ 0, & r > R_0 + \Delta R \end{cases}.$$

Obviously  $f_\Lambda$  is continuous since  $f_\Lambda(r) = 0$  already on  $(R_0, R_0 + \Delta R)$  but  $\frac{d}{dr}f_\Lambda(r, v)$  is not continuous in general.

Via  $\mu_\Lambda = \ln(E_0) - y_\Lambda$  and

$$(3.29) \quad e^{-2\lambda_\Lambda} = 1 - \frac{r^2\Lambda}{3} - \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, y_\Lambda(s)) ds$$

one can construct a local solution  $\mu_\Lambda, \lambda_\Lambda \in C^2([0, R_c])$  of (2.3), (2.4), where  $R_c > R_0 + \Delta R$ . This solution fulfills the boundary conditions  $\lambda_\Lambda(0) = 0$ ,  $\mu_\Lambda(0) = \ln(E_0) - y_0$ ,  $\lambda'_\Lambda(0) = \mu'_\Lambda(0) = 0$ . We now see that  $E_0 = e^{\mu(R_0, \Lambda)}$  and continue  $\mu_\Lambda$  and  $\lambda_\Lambda$  with the Schwarzschild-deSitter coefficients  $\mu_S, \lambda_S$  given by

$$(3.30) \quad e^{2\mu_S} = e^{-2\lambda_S} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}$$

in a continuous way beyond  $R_0 + \Delta R$ . From equation (3.26) we deduce that also the derivatives of  $\mu_\Lambda$  and  $\mu_S$  can be glued together in a continuous way. The functions  $\mu_\Lambda, \lambda_\Lambda$ , and  $f_\Lambda$  solve the Einstein-Vlasov system (2.2), (2.3), (2.4) globally.  $\square$

**Remark 3.8.** *In the isotropic case, i.e.  $L_0 = \ell = 0$  in the ansatz (2.19) for the distribution function  $f$ , the matter quantities  $\varrho$  and  $p$  are monotonically decreasing. This implies that their support in space is a ball. In the anisotropic case however, so called shell solutions occur, cf. [8]. The support of such matter shells is in general not connected.*

#### 4. STATIC, ANISOTROPIC, GLOBALLY REGULAR SOLUTIONS FOR $\Lambda < 0$

**4.1. Local existence.** In this section an existence lemma for  $\Lambda < 0$  is stated for small radii. This lemma corresponds to the first part of the proof of Theorem 2.2 in [24] for the case  $\Lambda = 0$ .

**Lemma 4.1.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) and let  $G_\phi, H_\phi$  be defined by equations (2.21) and (2.22), respectively. Then for every  $y_0 \in \mathbb{R}$  and every  $\Lambda < 0$  there exists a  $\delta > 0$  such that there exists a unique solution  $y_\Lambda \in C^2([0, \delta])$  of equation (2.24) with initial value  $y_\Lambda(0) = y_0$ .*

*Proof.* The proof works in an exact analogue way as in the case  $\Lambda > 0$ .  $\square$

**4.2. Globally regular solutions for  $\Lambda < 0$ .** For negative cosmological constants the global existence of solutions can be proved in an analogue way as done in [23] for the case  $\Lambda = 0$ . After establishing the local existence of solutions analog to the  $\Lambda > 0$  case, we show that the metric components stay bounded for all  $r \in \mathbb{R}_+$  with an energy estimate. This will yield the global existence of solutions of the Einstein-Vlasov system with negative cosmological constant. In the next step we show by virtue of a suitable choice of an ansatz for the matter distribution  $f$ , that the matter quantities  $\varrho$  and  $p$  are of bounded support. In the following theorem the existence on spatial intervals of the form  $\mathbb{R} \setminus [0, r_0)$ , for  $r_0 > 0$  is included for the purpose of applying the same theorem to the construction of static spacetimes with Schwarzschild singularities in the center (cf. Section 5.2). The solutions of interest here are those where the radius variable takes values in all of  $\mathbb{R}$ .

**Theorem 4.2.** *Let  $\Lambda < 0$  and let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) and let  $G_\phi$  and  $H_\phi$  be defined by equations (2.21) and (2.22). Then for every  $r_0 \geq 0$  and  $\mu_0, \lambda_0 \in \mathbb{R}$  there exists a unique solution  $\lambda_\Lambda, \mu_\Lambda \in C^1([r_0, \infty))$  of the Einstein-Vlasov system (2.2)-(2.6) with  $\mu_\Lambda(r_0) = \mu_0$  and  $\lambda_\Lambda(r_0) = \lambda_0$ . One has  $\lambda_0 = 0$  if  $r_0 = 0$ .*

*Proof.* We use an energy argument similar to [23]. Let  $y_\Lambda \in C^2([r_0, r_0 + \delta])$  be the local solution of equation (2.24) with  $y_\Lambda(r_0) = \ln(E_0)e^{-\mu_0}$ . If  $r_0 = 0$  the existence of this local solution is established by Lemma 4.1 and in the case  $r_0 > 0$  the existence of a local solution follows directly from the regularity of the right hand sides of (2.3) and (2.4). Let  $[r_0, R_c)$  be the maximal interval of existence of this solution. By  $\mu_\Lambda = \ln(E_0) - y_\Lambda$  and

$$(4.1) \quad e^{-2\lambda_\Lambda} = 1 - \frac{\Lambda}{3} \left( r^2 - \frac{r_0^3}{r} \right) - \frac{2}{r} \left( \frac{r_0}{2} (1 - e^{-2\lambda_0}) + 4\pi \int_{r_0}^r s^2 G_\phi(s, y_\Lambda(s)) ds \right)$$

one constructs a local solution  $\mu_\Lambda, \lambda_\Lambda \in C^2([r_0, R_c])$  of equations (2.3) and (2.4). We define

$$(4.2) \quad w_\Lambda(r) = -\frac{\Lambda}{12\pi} + \frac{1}{r^3} \left( -\frac{r_0^3 \Lambda}{24\pi} + \frac{r_0}{8\pi} (1 - e^{-2\lambda_0}) + \int_{r_0}^r s^2 \varrho_\Lambda(s) ds \right).$$

The Einstein equation (2.3) implies

$$(4.3) \quad \mu'_\Lambda(r) = 4\pi r e^{2\lambda_\Lambda(r)} (p_\Lambda(r) + w_\Lambda(r)).$$

By adding equations (2.3) and (2.4) we have

$$(4.4) \quad (\mu'_\Lambda(r) + \lambda'_\Lambda(r)) = 4\pi r e^{2\lambda_\Lambda(r)} (p_\Lambda(r) + \varrho_\Lambda(r)).$$

We assume  $R_c < \infty$  and consider the quantity  $e^{\mu_\Lambda + \lambda_\Lambda} (p_\Lambda + w_\Lambda)$  on the interval  $[\frac{R_c}{2}, R_c)$ . On this interval, in particular away from the origin, a differential inequality will be established that will allow us to deduce that both  $\mu_\Lambda$  and  $\lambda_\Lambda$  are bounded on  $[\frac{R_c}{2}, R_c)$ . Using the TOV equation (2.16) we obtain for  $r \in [\frac{R_c}{2}, R_c)$

$$(4.5) \quad \begin{aligned} \frac{d}{dr} \left( e^{\mu_\Lambda + \lambda_\Lambda} (p_\Lambda + w_\Lambda) \right) &= e^{\mu_\Lambda + \lambda_\Lambda} \left( -\frac{2p_\Lambda}{r} - \frac{3w_\Lambda}{r} - \frac{\Lambda}{4\pi r} + \frac{2p_{T\Lambda}}{r} + \frac{\varrho_\Lambda}{r} \right) \\ &\leq C_1 e^{\mu_\Lambda + \lambda_\Lambda} = \underbrace{\frac{C_1}{p_\Lambda + w_\Lambda}}_{=: C_2} (p_\Lambda + w_\Lambda) e^{\mu_\Lambda + \lambda_\Lambda}. \end{aligned}$$

In the course of this estimate we have used that  $\frac{\Lambda}{4\pi r}$ ,  $p_{T\Lambda}(r)/r$  and  $\varrho_\Lambda(r)/r$  stay bounded for  $r \in [\frac{R_c}{2}, R_c)$ . The constant  $C_2$  is bounded since  $w_\Lambda(r) > 0$  for negative  $\Lambda$ . It follows

$$(4.6) \quad \frac{d}{dr} \ln \left( e^{\mu_\Lambda + \lambda_\Lambda} (p_\Lambda + w_\Lambda) \right) \leq C_2 \quad \Rightarrow \quad \lambda_\Lambda + \mu_\Lambda < \infty.$$

Equation (4.3) implies that  $\mu'_\Lambda(r) \geq 0$  and therefore  $\mu_\Lambda(r) \geq \mu_0$ . We also have

$$(4.7) \quad e^{-2\lambda_\Lambda} \leq 1 + \frac{r^2 |\Lambda|}{3} \leq \frac{3 + R_c^2 |\Lambda|}{3} < \infty.$$

This in turn implies  $\lambda_\Lambda > -\infty$  and we deduce from equation (4.6) that both  $\mu_\Lambda$  and  $\lambda_\Lambda$  are bounded on  $[\frac{R_c}{2}, R_c)$ . This allows to continue  $\mu_\Lambda$  and  $\lambda_\Lambda$  as  $C^2$ -solutions of the Einstein equations beyond  $R_c$  which contradicts its definition. So  $R_c = \infty$ .  $\square$

We prove in the following theorem that the distribution function in the previous theorem is compactly supported which yield physically reasonable solutions.

**Theorem 4.3.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) and let  $\mu_0 \in \mathbb{R}$ ,  $r_0 \geq 0$  and let  $\lambda_\Lambda, \mu_\Lambda \in C^1([r_0, \infty))$ ,  $f(x, v) = \Phi(E, L)$  be the unique global-in- $r$  solution of the Einstein-Vlasov system (2.2) – (2.6) with negative cosmological constant where  $\mu_\Lambda(0) = \mu_0$  such*

that  $y_0 = \ln(E_0)e^{-\mu_0} > 0$ . Then there exists  $R_0 \in (r_0, \infty)$  such that the spatial support of  $f_\Lambda$  is contained in the interval  $[r_0, R_0]$ .

*Proof.* Due to Lemma 2.3, (iii) we have vacuum if  $y_\Lambda(r) \leq 0$ . By assumption we have  $y_\Lambda(0) > 0$ . In the following we show that  $\lim_{r \rightarrow \infty} y_\Lambda(r) < 0$ . Since  $y_\Lambda$  is continuous and monotonically decreasing, this implies that  $y_\Lambda$  possesses a single zero  $R_0$  and the support of the matter quantities  $\varrho_\Lambda$  and  $p_\Lambda$  is contained in  $[0, R_0]$ .

We define  $y_{\text{vac},\Lambda}$  by

$$(4.8) \quad y_{\text{vac},\Lambda} = y_0 - \frac{1}{2} \ln \left( 1 - \frac{r^2 \Lambda}{3} \right).$$

So we have

$$(4.9) \quad y'_{\text{vac},\Lambda}(r) = -\frac{\kappa/2}{1 - \frac{\Lambda r^2}{3}} \left( -\frac{2r\Lambda}{3\kappa} \right)$$

and  $y_{\text{vac},\Lambda}(0) = y_\Lambda(0) = y_0$ . Furthermore, since  $y'_\Lambda(r) < y'_{\text{vac},\Lambda}(r)$  which can be seen immediately by means of equation (2.24) we have

$$(4.10) \quad y_\Lambda(r) \leq y_{\text{vac},\Lambda}(r) = y_0 - \frac{1}{2} \ln \left( 1 + \frac{r^2 |\Lambda|}{3} \right) \xrightarrow{r \rightarrow \infty} -\infty < 0$$

and the theorem follows.  $\square$

**Remark 4.4.** *The solution coincides with Schwarzschild-AdS for  $r \geq R_0$  if the continuity condition*

$$(4.11) \quad \mu_\Lambda(R_0) = \ln(E_0) - y_\Lambda(R_0) = \frac{1}{2} \ln \left( 1 - \frac{R_0^2 \Lambda}{3} - \frac{2M}{R_0} \right)$$

is fulfilled, where  $M = 4\pi \int_0^{R_0} s^2 \varrho_\Lambda(s) ds$ . So if  $y_0$  is given, the corresponding value of  $E_0$  in the ansatz  $\Phi$  for the matter distribution  $f$  can be read off.

## 5. SOLUTIONS WITH A SCHWARZSCHILD SINGULARITY AT THE CENTER

In this section we construct spherically symmetric, static solutions of the Einstein-Vlasov system with non-vanishing cosmological constant that contain a Schwarzschild singularity at the center. We consider both the case with a positive and a negative cosmological constant. The construction for the case  $\Lambda > 0$  makes use of the corresponding solutions with vanishing  $\Lambda$ . In the following we will call this solution, where  $\Lambda = 0$ , a *background solution*. The global existence of the background solution is proved in [23]. The matter quantities belonging to this background solution are of finite support.

**5.1. Matter shells immersed in Schwarzschild-deSitter spacetime.** The construction of the solution with  $\Lambda > 0$  can be outlined as follows. In the vacuum case, i.e. when the right hand sides of the Einstein equations (2.3) and (2.4) are zero, the solutions are given by

$$(5.1) \quad e^{2\mu(r)} = 1 - \frac{r^2 \Lambda}{3} - \frac{2M_0}{r}, \quad e^{2\lambda(r)} = \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M_0}{r} \right)^{-1}, \quad r > r_{B\Lambda}$$

where  $r_{B\Lambda}$  is defined to be the black hole event horizon, i.e. the smallest positive zero of  $1 - r^2 \Lambda/3 - 2M_0/r$ . If one chooses  $L_0$  and  $M_0$  appropriately and  $\Lambda$  sufficiently small the following configuration is on hand. For small  $r > r_{B\Lambda}$  one sets  $f(x, v) \equiv 0$  and the metric is given by Schwarzschild-deSitter. Thus one has the coefficients (5.1). Increasing the radius  $r$  one reaches an interval  $[r_{-\Lambda}, r_{+\Lambda}]$  where also an ansatz  $f(x, v) = \Phi(E, L)$  of the form (2.19) yields vacuum, i.e.  $G_\phi(r, y(r)) = H_\phi(r, y(r)) = 0$ . In this interval it is possible to glue to the Schwarzschild-deSitter solution (5.1) a non vacuum solution solving the Einstein-Vlasov system. It will be shown that the matter quantities  $\varrho_\Lambda$  and  $p_\Lambda$  of this

solution have finite support. Beyond the support of the matter quantities the solution will be continued again by Schwarzschild-deSitter.

For negative cosmological constant, globally defined solutions can be constructed as well. Like in the case above, the black hole is surrounded by a vacuum shell which is on its part surrounded by a shell containing matter. In the outer region, we again have vacuum.

Before we consider the system with  $\Lambda \neq 0$  we establish a Buchdahl type inequality for solutions of the Einstein equations with a Schwarzschild singularity at the center. This inequality is relevant for the proof of existence of solutions of the Einstein-Vlasov system with  $\Lambda > 0$ .

**Lemma 5.1.** *Let  $\lambda, \mu \in C^1([0, \infty))$  and let  $\varrho, p, p_T \in C^0([0, \infty))$  be functions that satisfy the system of equations (2.3-2.7) with a Schwarzschild singularity with mass parameter  $M_0 > 0$  at the center, and such that  $p + 2p_T \leq \varrho$ . Then the inequality*

$$(5.2) \quad \frac{2(M_0 + m(r))}{r} \leq \frac{8}{9}$$

holds for all  $r \in [\frac{9}{4}M_0, \infty)$  where  $m(r)$  is given by

$$(5.3) \quad m(r) = 4\pi \int_{2M_0}^r s^2 \varrho(s) ds.$$

*Proof.* For the prove of the lemma we apply techniques that are already used in [19] to prove the Buchdahl inequality for globally regular solutions without Schwarzschild singularity. Only the steps that differ from the proof of Theorem 4.1 in [19], or Theorem 1 in [5] for the charged case, will be described in detail.

By integrating the Einstein equation (2.3) over the interval  $(\frac{9M_0}{4}, r)$  we obtain

$$(5.4) \quad e^{-2\lambda} = 1 - \frac{9M_0}{4r} (1 - e^{-2\lambda_0}) - \frac{8\pi}{r} \int_{\frac{9M_0}{4}}^r s^2 \varrho(s) ds,$$

where  $\lambda_0 = \lambda(\frac{9M_0}{4})$ . Since we have vacuum on  $(2M_0, \frac{9M_0}{4})$  on this interval the metric is given by the Schwarzschild metric and one can compute  $\lambda_0$  explicitly. One finds that

$$(5.5) \quad e^{-2\lambda} = 1 - \frac{2(M_0 + m(r))}{r}.$$

We plug this into the other Einstein equation (2.4) and obtain the differential equation

$$(5.6) \quad \mu'(r) = \frac{1}{1 - \frac{2(M_0 + m(r))}{r}} \left( 4\pi r p + \frac{M_0 + m(r)}{r^2} \right).$$

We now introduce the variables

$$(5.7) \quad x = \frac{2(M_0 + m(r))}{r}, \quad y = 8\pi r^2 p(r).$$

Note that  $x < 1$  and  $y \geq 0$ . The first inequality must hold true since otherwise the metric function  $\lambda$  would not stay bounded. Next we let  $\beta = 2\ln(r)$  and consider the curve  $(x(e^{\beta/2}), y(e^{\beta/2}))$  parameterized by  $\beta$  in  $[0, 1) \times [0, \infty)$ . In the following a dot denotes the derivative with respect to  $\beta$ . Using the Einstein equations and the generalized TOV equation (2.16) one checks that  $x$  and  $y$  satisfy the equations

$$(5.8) \quad 8\pi r^2 \varrho = 2\dot{x} + x,$$

$$(5.9) \quad 8\pi r^2 p = y,$$

$$(5.10) \quad 8\pi r^2 p_T = \frac{x + y}{2(1 - x)} \dot{x} + \dot{y} + \frac{(x + y)^2}{4(1 - x)}.$$

By virtue of these equations (5.8) – (5.10) the condition  $p + 2p_T \leq \varrho$  can be written in the form

$$(5.11) \quad (3x - 2 + y)\dot{x} + 2(1 - x)\dot{y} \leq -\frac{\alpha(x, y)}{2}, \quad \alpha = 3x^2 - 2x + y^2 + 2y.$$

From now on the proof is analogue to the proof of Theorem 1 in [5] for the charged case. One defines the quantity

$$(5.12) \quad w(x, y) = \frac{(3(1 - x) + 1 + y)^2}{1 - x}$$

and shows that since  $0 \leq x < 1$  and  $y \leq 0$  this quantity is bounded by 16 along the curve  $(x, y)$  with an optimization procedure. The inequality  $w \leq 16$  is already equivalent to

$$(5.13) \quad \frac{2(M_0 + m(r))}{r} \leq \frac{8}{9}$$

for all  $r \in [\frac{9M_0}{4}, \infty)$  and the proof is complete.  $\square$

**Remark 5.2.** *In the case when  $M_0 = 0$  it is known that the inequality is sharp, cf. [4] and [19]. For the purpose of this work the bound (5.2) is sufficient and we have not tried to show sharpness.*

In the course of the proof of Theorem 5.5 we will need a continuation criterion for the solution of the Einstein equations, namely the following statement.

**Lemma 5.3.** *Let  $\Lambda > 0$ ,  $\mu_0 \in \mathbb{R}$  and  $M_0, r_0 > 0$ . Let  $G_\phi$  and  $H_\phi$  defined by equations (2.21) and (2.22). Then the equation*

$$(5.14) \quad \mu'_\Lambda = \frac{1}{1 - \frac{\Lambda}{3} \left( r^2 - \frac{r_0^3}{r} \right) - \frac{2}{r} \left( M_0 + 4\pi \int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s)) ds \right)} \\ \times \left( 4\pi r H_\phi(s, \mu_\Lambda(s)) - \Lambda \left( \frac{r}{3} + \frac{r_0^3}{6r^2} \right) \frac{1}{r^2} \left( M_0 + 4\pi \int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s)) ds \right) \right)$$

has a unique local  $C^2$ -solution  $\mu_\Lambda$  with  $\mu(r_0) = \mu_0$  with maximal interval of existence  $[r_0, R_c)$ ,  $R_c > 0$ . Moreover, there exists  $R_D \leq R_c$  such that

$$(5.15) \quad \liminf_{r \rightarrow R_D} \left( 1 - \frac{\Lambda}{3} \left( r^2 - \frac{r_0^3}{r} \right) - \frac{2}{r} \left( M_0 + 4\pi \int_{r_0}^r s^2 G_\phi(s, \mu_\Lambda(s)) ds \right) \right) = 0.$$

*Proof.* The local existence of a  $C^2$ -solution of equation (5.14) follows from the regularity of the right hand side. Basically one is in the situation of Lemma 3.2, i.e., the case with a regular center and  $\Lambda > 0$ , except for the fact that there are additional terms containing  $r_0$  and  $M_0$ . But on a finite interval  $[r_0, R_c)$  these terms are bounded and well behaved, i.e. the proof can be carried out in an analogue way.  $\square$

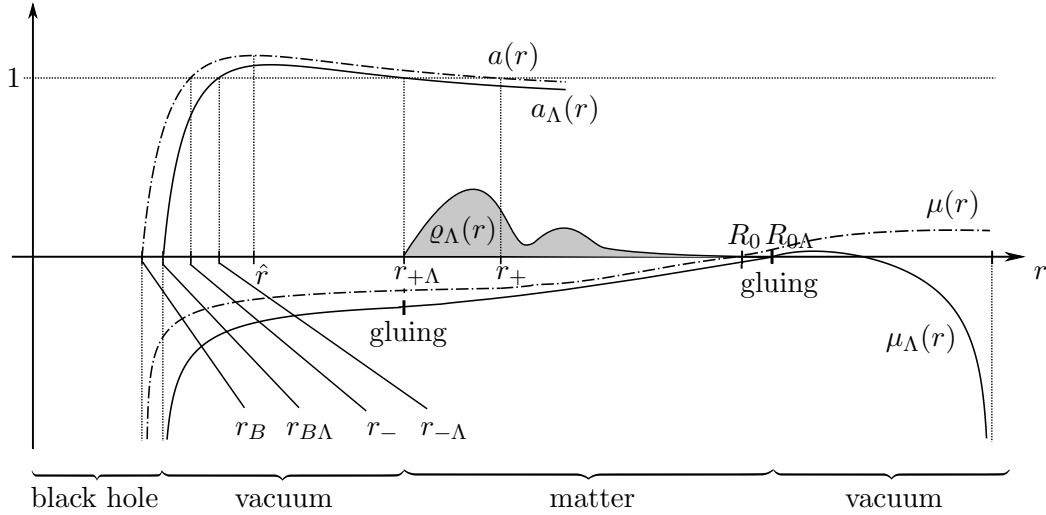
**Remark 5.4.** *Lemma 5.3 implies that if the denominator of the right hand side of equation (5.14) is strictly larger than zero on an interval  $[r_0, r)$ , then  $\mu_\Lambda$  can be extended beyond  $r$  as a solution of (5.14).*

The following theorem states the existence of solutions for  $\Lambda > 0$  with a Schwarzschild singularity at the center.

**Theorem 5.5.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) with  $E_0 = 1$  and let  $L_0, M_0 \geq 0$  such that  $L_0 > 16M_0^2$ . Then there exists a unique solution  $\mu_\Lambda, \lambda_\Lambda \in C^2((r_{B\Lambda}, \infty))$ ,  $f \in C^0((r_{B\Lambda}, \infty) \times \mathbb{R}^3)$  of the Einstein-Vlasov system (2.2) – (2.6) for  $\Lambda > 0$  sufficiently small. The spatial support of the distribution function  $f_\Lambda$  is contained in a shell  $\{r_{+\Lambda} < r < R_{0\Lambda}\}$ . In the complement of this shell the solution of the Einstein equations is given by the Schwarzschild-deSitter metric.*

**Remark 5.6.** *In the course of the proof one will come across the fact that in one of the vacuum regions, either  $r \leq r_{+\Lambda}$  or  $r \geq R_{0\Lambda}$ , the component  $\mu_{\text{vac}}$  given by  $e^{2\mu_{\text{vac}}} = 1 - \frac{r^2\Lambda}{3} - \frac{2M}{r}$  of the Schwarzschild-deSitter metric will be shifted by a constant. But this shift is just a reparametrization of the time  $t$  [23]. Thus the shell of Vlasov matter causes a redshift.*

*Proof.* In the first part of the proof we consider the black hole region and show that the chosen parameters lead to the configuration depicted in Figure 1. Then we make use of the existence of a background solution and construct the desired solution  $\mu_\Lambda$ .



**Figure 1** – Qualitative sketch of a black hole configuration surrounded by a shell of matter

We define the functions

$$(5.16) \quad a(r) = \sqrt{1 - \frac{2M_0}{r}} \sqrt{1 + \frac{L_0}{r^2}},$$

$$(5.17) \quad a_\Lambda(r) = \sqrt{1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}} \sqrt{1 + \frac{L_0}{r^2}}.$$

Moreover, we define  $r_-$  and  $r_+$  to be the first and second radius where  $a(r) = 1$ , respectively, and  $r_B := 2M_0$  to be the event horizon of the black hole. Since  $L_0 > 16M_0^2$  we have  $r_B < r_- < r_+$  (cf. [23]). Note also that  $r_+ > 4M_0 > \frac{18}{5}M_0$ . Since  $9M_0^2\Lambda < 1$  by assumption ( $\Lambda$  is chosen to be small), there exists a black hole horizon  $r_{B\Lambda}$  of the Schwarzschild-deSitter metric with parameters  $M_0$  and  $\Lambda$ . It can be calculated explicitly<sup>2</sup> by

$$(5.18) \quad r_{B\Lambda} = -\frac{2}{\sqrt{\Lambda}} \cos \left( \frac{1}{3} \arccos \left( -3M_0\sqrt{\Lambda} \right) + \frac{\pi}{3} \right).$$

<sup>2</sup>To assure oneself of that one has chosen the right zero, using  $\frac{d}{dx} \arccos(x) = -\frac{1}{\sqrt{1-x^2}}$  one checks

$$\lim_{\Lambda \rightarrow 0} r_{B\Lambda} \stackrel{\text{L'Hôpital}}{=} \lim_{\Lambda \rightarrow 0} \frac{2 \sin \left( \frac{1}{3} \arccos \left( -3M_0\sqrt{\Lambda} \right) + \frac{\pi}{3} \right) \frac{1}{3\sqrt{1-(3M_0\sqrt{\Lambda})^2}} \frac{3M_0}{2\sqrt{\Lambda}}}{1/(2\sqrt{\Lambda})} = 2M_0.$$



Note that  $r_B < r_{B\Lambda}$ . We construct an upper bound to  $r_{B\Lambda}$ . Set  $v(r) = 1 - \frac{2M_0}{r}$ .

$$\begin{aligned}
 (5.19) \quad v(r_{B\Lambda}) &= \int_{r_B}^{r_{B\Lambda}} v'(s) ds + \underbrace{v(r_B)}_{=0} \\
 &\geq \int_{r_B}^{r_{B\Lambda}} \left( \inf_{s \in [r_B, r_{B\Lambda}]} v'(s) \right) ds = (r_{B\Lambda} - r_B) v'(r_{B\Lambda}) \\
 \Rightarrow \quad r_{B\Lambda} &\leq r_B + \frac{v(r_{B\Lambda})}{v'(r_{B\Lambda})}
 \end{aligned}$$

A short calculation yields  $v(r_{B\Lambda}) = \frac{r_{B\Lambda}^2 \Lambda}{3}$  and  $v'(r_{B\Lambda}) = \frac{2M_0}{r_{B\Lambda}^2}$ . One also checks by explicit calculation that  $\frac{dr_{B\Lambda}}{d\Lambda} > 0$ . So the distance

$$(5.20) \quad r_{B\Lambda} - r_B \leq \frac{r_{B\Lambda}^4 \Lambda}{6M_0}$$

between the two horizons can be made arbitrarily small if  $\Lambda$  is chosen to be sufficiently small. In particular we need  $\Lambda$  to be small enough to assure  $r_{B\Lambda} < r_-$ .

Next we define  $r_{-\Lambda}$  and  $r_{+\Lambda}$  to be the first and second radius where  $a_\Lambda(r) = 1$ . Note that  $a(r) > a_\Lambda(r)$  for all  $r \in (r_{B\Lambda}, r_C)$ , where  $r_C$  is the cosmological horizon of the vacuum solution, thus the second positive zero of  $1 - r^2 \Lambda / 3 - 2M_0/r$ . Between  $r_-$  and  $r_+$  the function  $a(r)$  has a unique maximum at  $r = \hat{r}$ , given by

$$(5.21) \quad \hat{r} = \frac{L_0 - \sqrt{L_0^2 - 12M_0^2 L_0}}{2M_0}.$$

We consider the distance between  $a^2(r)$  and  $a_\Lambda^2(r)$  at this radius  $\hat{r}$ :

$$(5.22) \quad |a^2(\hat{r}) - a_\Lambda^2(\hat{r})| = \Lambda \frac{\hat{r}^2 + L_0}{3}.$$

Choosing  $\Lambda$  sufficiently small one can attain  $|a^2(\hat{r}) - a_\Lambda^2(\hat{r})| < a^2(\hat{r}) - 1$ . This implies that  $a_\Lambda(r) - 1$  has exactly two zeros in the interval  $(r_-, r_+)$ . This in turn yields the desired configuration

$$(5.23) \quad 2M_0 = r_B < r_{B\Lambda} < r_- < r_{-\Lambda} < \hat{r} < r_{+\Lambda} < r_+.$$

In the vacuum region  $[r_{-\Lambda}, r_{+\Lambda}]$  the function  $a_\Lambda(r)$  coincides with the expression  $e^{-y_\Lambda(r)} \sqrt{1 + \frac{L_0}{r^2}}$ . Lemma 2.3, (iii) implies that therefore for  $r \in [r_{-\Lambda}, r_{+\Lambda}]$  also the ansatz  $\Phi$  for the distribution function  $f$  yields  $\varrho_\Lambda(r) = G_\phi(r, y_\Lambda(r)) = 0$  and  $p_\Lambda(r) = H_\phi(r, y_\Lambda(r)) = 0$ . So at  $r = r_{+\Lambda}$  one can continue  $f$  by the ansatz  $\Phi$  in a continuous way and for  $r \geq r_{+\Lambda}$  the Einstein equations lead to the differential equation

$$\begin{aligned}
 (5.24) \quad \mu'_\Lambda &= \frac{1}{1 - \frac{\Lambda}{3} \left( r^2 - \frac{r_{+\Lambda}^3}{r} \right) - \frac{2}{r} \left( \frac{r_{+\Lambda}}{2} (1 - e^{-2\lambda_0}) + 4\pi \int_{r_{+\Lambda}}^r s^2 \varrho_\Lambda(s) ds \right)} \\
 &\quad \times \left( 4\pi r p_\Lambda - \Lambda \left( \frac{r}{3} + \frac{r_{+\Lambda}^3}{6r^2} \right) + \frac{r_{+\Lambda}}{2r^2} (1 - e^{-2\lambda_0}) + \frac{4\pi}{r^2} \int_{r_{+\Lambda}}^r s^2 \varrho_\Lambda(s) ds \right)
 \end{aligned}$$

where  $\lambda_0 = \lambda(r_{+\Lambda})$ .

There exists a background solution  $\mu \in C^2((2M_0, \infty))$  to the Einstein equations with  $\Lambda = 0$  (cf. [23]). For  $r \in (2M_0, r_{+\Lambda})$  this solution is given by the Schwarzschild metric and for  $r > r_{+\Lambda}$  as a solution of equation (5.24) with  $\Lambda = 0$ . The background solution is continuous at  $r_{+\Lambda}$  if

$$(5.25) \quad \frac{r_{+\Lambda}}{2} (1 - e^{-2\lambda_0}) = M_0.$$

Furthermore, the background solution  $\mu$  has the property that there exists  $R_0 > 0$  such that  $\mu(R_0) = 0$  which implies that the support of matter quantities  $\varrho$  and  $p$  is contained in the interval  $(r_+, R_0)$  (cf. [23]). In the remainder of the proof we show that using properties of this background solution  $\mu$  one obtains a global solution  $\mu_\Lambda$  of equation (5.24). We set

$$(5.26) \quad \mu_{0\Lambda} = \frac{1}{2} \ln \left( 1 - \frac{r_{+\Lambda}^2 \Lambda}{3} - \frac{2M_0}{r_{+\Lambda}} \right),$$

$$(5.27) \quad \mu_0 = \mu(r_{+\Lambda}) = \frac{1}{2} \ln \left( 1 - \frac{2M_0}{r_{+\Lambda}} \right).$$

In the following we seek for a solution  $\mu_\Lambda$  of equation (5.24) on an interval beginning at  $r = r_{+\Lambda}$  with the initial value  $\mu_{0\Lambda}$  at given in (5.26) that we can glue to the vacuum solution on  $(r_{B\Lambda}, r_{+\Lambda}]$ . Note that  $\mu_{0\Lambda} < 0$ . Since there are no issues with an irregular center the local existence of  $\mu_\Lambda$  on an interval  $(r_{+\Lambda}, r_{+\Lambda} + \delta]$ ,  $\delta > 0$  follows from the regularity of the right hand side of equation (5.24). So let  $(2M_0, R_c)$  be the maximum interval of existence of  $\mu_\Lambda$ . We define

$$(5.28) \quad v_{M_0}(r) = 1 - \frac{2}{r} \left( M_0 + 4\pi \int_{r_{+\Lambda}}^r s^2 \varrho(s) ds \right),$$

$$(5.29) \quad v_{M_0\Lambda}(r) = 1 - \frac{\Lambda}{3} \left( r^2 - \frac{r_{+\Lambda}^3}{r} \right) - \frac{2}{r} \left( M_0 + 4\pi \int_{r_{+\Lambda}}^r s^2 \varrho_\Lambda(s) ds \right)$$

as the denominator of the right hand side of equation (5.24). We set

$$(5.30) \quad \Delta v_0 := \frac{1}{18} v_{M_0\Lambda}(r_{+\Lambda}) = \frac{1 - \frac{2M_0}{r_{+\Lambda}}}{18} \leq \frac{1}{18}$$

and define the radii

$$(5.31) \quad \begin{aligned} r^* &= \inf \{ r \in (r_{+\Lambda}, R_c) \mid v_{M_0\Lambda}(r) = \Delta v_0 \}, \\ \tilde{r} &= \sup \{ r \in (r_{+\Lambda}, R_c) \mid |\mu_\Lambda(r) - \mu(r)| \leq \mu(R_0 + \Delta R) \}, \end{aligned}$$

and set  $\tilde{r}^* := \min\{\tilde{r}, r^*\}$ . Note that  $\mu(R_0 + \Delta R) > 0$  since  $\mu(R_0) = 0$  and  $\mu$  is strictly increasing. We assume that  $r \leq \tilde{r}^*$  and calculate  $|\mu(r) - \mu_\Lambda(r)|$ . To make calculations more convenient, we extend  $\varrho$  and  $p$  on  $[0, 2M_0]$  as constant zero such that integrals of  $\varrho$  and  $p$  over  $(r_+, r)$  can be replaced by integrals over  $(0, r)$ . First we calculate

$$(5.32) \quad |\mu_0 - \mu_{0\Lambda}| = \frac{1}{2} \ln \left[ 1 + \frac{r_{+\Lambda}^2 \Lambda}{3} \left( 1 - \frac{r_{+\Lambda}^2 \Lambda}{3} - \frac{2M_0}{r_{+\Lambda}} \right)^{-1} \right] =: C_{0\Lambda}(r).$$

We write

$$(5.33) \quad \begin{aligned} &|\mu(r) - \mu_\Lambda(r)| \\ &\leq \int_{r_{+\Lambda}}^r \frac{1}{v_{M_0\Lambda}(s)} \left[ 4\pi s |p_\Lambda(s) - p(s)| - \Lambda \left( \frac{s}{3} + \frac{r_{+\Lambda}^3}{s^2} \right) \right. \\ &\quad \left. + \frac{4\pi}{s^2} \int_0^s \sigma^2 |\varrho_\Lambda(\sigma) - \varrho(\sigma)| d\sigma \right] ds \\ &\quad + \int_{r_{+\Lambda}}^r \left( 4\pi s p(s) + \frac{4\pi}{s^2} \int_0^s \sigma^2 \varrho(\sigma) d\sigma \right) \left| \frac{1}{v_{M_0\Lambda}(s)} - \frac{1}{v_{M_0}(s)} \right| ds + C_{0\Lambda}(r) \end{aligned}$$

We would like to apply the generalized Buchdahl inequality (Lemma 5.1) to the background solution  $\mu$  on the interval  $[r_{+\Lambda}, \infty)$ . We have that  $r_{+\Lambda} > \hat{r} \geq 3M_0 > 9/4M_0$ . The crucial condition is the existence of a vacuum region on  $(2M_0, \frac{9}{4}M_0]$ . But this is

ensured by virtue of the assumption  $L_0 > 16M_0^2$  which implies  $r_+ > 4M_0$ . So the difference  $|\mu(r) - \mu_\Lambda(r)|$  can be further simplified and estimated. Using similar estimates as in Appendix B we obtain an inequality of the form

$$(5.34) \quad |\mu(r) - \mu_\Lambda(r)| \leq C_\Lambda(r) + C(r) \int_0^r (|p(s) - p_\Lambda(s)| + |\varrho(s) - \varrho_\Lambda(s)|) ds$$

where  $C(r)$  is increasing in  $r$ ,  $C_\Lambda(r)$  is increasing both in  $\Lambda$  and  $r$  and we have  $C_\Lambda(r) = 0$  if  $\Lambda = 0$ . Note that the constants are fully determined by  $M_0$ ,  $L_0$ ,  $\phi$  and  $\mu$ .

In virtue of the mean value theorem, the sum  $|p_\Lambda - p| + |\varrho_\Lambda - \varrho|$  can be estimated as

$$(5.35) \quad |p_\Lambda(r) - p(r)| + |\varrho_\Lambda(r) - \varrho(r)| \leq C \cdot |\mu_\Lambda(r) - \mu(r)|,$$

where the constant  $C$  is determined by the derivatives of  $G_\phi$  and  $H_\phi$ . A Grönwall argument yields  $|\mu_\Lambda(r) - \mu(r)| \leq C_{\mu\Lambda}(r)$  implying  $|\varrho_\Lambda(r) - \varrho(r)| \leq C_{g\Lambda}(r)$  with certain constants  $C_{g\Lambda}$  and  $C_{\mu\Lambda}$ .

One can choose  $\Lambda$  small enough such that for all  $r \in (r_{+\Lambda}, R_0 + \Delta R]$  we have

$$(5.36) \quad |\mu_\Lambda(r) - \mu(r)| < \mu(R_0 + \Delta R).$$

Moreover, we consider the difference

$$(5.37) \quad |v_{M_0}(r) - v_{M_0\Lambda}(r)| \leq \frac{\Lambda}{3} \left| r^2 - \frac{r_{+\Lambda}^3}{r} \right| + \frac{8\pi r^2}{3} C_{g\Lambda}(r).$$

Lemma 5.1 implies  $v_{M_0}(r) \geq \frac{1}{9}$  for all  $r \in (r_{+\Lambda}, \infty)$ . Choosing  $\Lambda$  sufficiently small, such that for all  $r \in (r_{+\Lambda}, R_0 + \Delta R]$  we have  $|v_{M_0}(r) - v_{M_0\Lambda}(r)| \leq \frac{1}{18}$  one obtains  $v_{M_0\Lambda} \geq \frac{1}{18}$  on  $(r_{+\Lambda}, R_0 + \Delta R]$ .

So altogether, one has deduced that  $\tilde{r}^* \geq R_0 + \Delta R$  if  $\Lambda$  is chosen sufficiently small. This implies that  $\mu_\Lambda$  exists at least on  $[0, R_0 + \Delta R]$  by Lemma 5.3 and also that  $\mu_\Lambda(R_0 + \Delta R) > 0$ . From the latter property one deduces that there exists a radius  $R_{0\Lambda} > R_0$  such that for all  $r \in [R_{0\Lambda}, R_0 + \Delta R]$  we have  $\varrho_\Lambda(r) = p_\Lambda(r) = 0$ . On this interval, we can glue an appropriately shifted Schwarzschild-de Sitter metric to  $\mu_\Lambda$ . This yields the desired solution defined on  $(r_{B\Lambda}, \infty)$ .  $\square$

**Remark 5.7.** *To see that the solutions constructed in Theorem 5.5 are non-vacuum, one checks that for  $r \geq r_{+\Lambda}$  one has*

$$(5.38) \quad \frac{d}{dr} a_\Lambda(r) < 0 \quad \text{and} \quad \frac{d^2}{dr^2} a_\Lambda(r) \leq 0.$$

Since  $a_\Lambda(r)$  corresponds to  $e^{-y_\Lambda(r)}$ , this implies that for some  $r > r_{+\Lambda}$  the quantity  $e^{-y_\Lambda(r)} \sqrt{1 + \frac{L_0}{r^2}} < 1$  which in turn implies by Lemma 2.3, (iii) that  $\varrho_\Lambda(r), p_\Lambda(r) > 0$  for some  $r > r_{+\Lambda}$ .

**Remark 5.8.** *In contrary to the metric without a singularity at the center, the metric with a Schwarzschild singularity does not coincide with the not shifted Schwarzschild-de Sitter solution for  $r > R_{0\Lambda}$ . This can be seen as follows. We have*

$$(5.39) \quad \mu'_\Lambda(r) \geq \frac{1}{2} \frac{d}{dr} \ln \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M_0}{r} \right).$$

Certainly, the mass parameter  $M$  of the vacuum solution, that is glued on in the outer region, is larger than  $M_0$ . This implies

$$(5.40) \quad 1 - \frac{r^2 \Lambda}{3} - \frac{2M_0}{r} > 1 - \frac{r^2 \Lambda}{3} - \frac{2M}{r}$$

for all  $r \in (r_{B\Lambda}, r_C)$ . So there is no ansatz  $\Phi$  for the matter distribution that yields a metric component  $\mu_\Lambda$  that connects the two vacuum solutions without any shift. But by suitable choice of  $\Phi$  and  $E_0$  one can determine whether the inner or the outer Schwarzschild-deSitter metric is shifted. For the maximal  $C^2$ -extension of the metric constructed in Theorem 5.5 we will need the solution to coincide with the not shifted Schwarzschild-deSitter metric for  $r > R_{0\Lambda}$ .

**5.2. Matter shells immersed in Schwarzschild-AdS spacetimes.** We construct solutions of the Einstein-Vlasov system with a Schwarzschild singularity at the center for the case  $\Lambda < 0$ . The result is given in the following theorem.

**Theorem 5.9.** *Let  $\Phi : \mathbb{R}^2 \rightarrow [0, \infty)$  be of the form (2.19) and let  $L_0, M_0 \geq 0$  such that  $L_0 < 16M_0^2$ . Then there exists a unique solution  $\mu_\Lambda, \lambda_\Lambda \in C^2((r_{B\Lambda}, \infty))$ ,  $f \in C^0((r_{B\Lambda}, \infty) \times \mathbb{R}^3)$  of the Einstein-Vlasov system (2.2) – (2.6) for  $\Lambda < 0$  and  $|\Lambda|$  sufficiently small. The spatial support of the distribution function  $f_\Lambda$  is contained in a shell,  $\{r_{+\Lambda} < r < R_{0\Lambda}\}$ . In the complement of this shell, the solution of the Einstein equations is given by the Schwarzschild-AdS metric.*

*Proof.* We define  $r_B := 2M_0$  to be the Schwarzschild black hole horizon of the background solution and  $r_{B\Lambda}$  to be the black hole horizon for the Schwarzschild-AdS with  $\Lambda < 0$ , i.e. the smallest positive zero of  $1 - r^2\Lambda/3 - 2M_0/r$ . Define also the functions

$$(5.41) \quad a(r) = \sqrt{1 - \frac{2M_0}{r}} \sqrt{1 + \frac{L_0}{r^2}},$$

$$(5.42) \quad a_\Lambda(r) = \sqrt{1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r}} \sqrt{1 + \frac{L_0}{r^2}}.$$

Moreover we define  $r_-$  and  $r_+$  to be the first and second positive zero of  $a(r) - 1$ , respectively, as well as  $r_{-\Lambda}$  and  $r_{+\Lambda}$  to be the first and second positive zero of  $a_\Lambda(r) - 1$ . The assumption  $L_0 < 16M_0^2$  assures that  $r_B < r_- < r_+$  but a priori  $r_{+\Lambda} = \infty$  and  $r_{-\Lambda} = \infty$  are possible. However, we show that the configuration is

$$(5.43) \quad r_{B\Lambda} < r_{-\Lambda} < r_{+\Lambda} < \infty.$$

First, we observe that  $a(r) < 1$  for all  $r > r_+$  and also that  $a_\Lambda(r) > a(r)$  for all  $r \in \mathbb{R}_+$  since  $\Lambda < 0$ . So we have  $r_{B\Lambda} < r_{-\Lambda} < r_- < r_+ < r_{+\Lambda}$ . It remains to show that  $r_{+\Lambda} < \infty$ . This is done by showing that for  $|\Lambda|$  sufficiently small the functions  $a$  and  $a_\Lambda$  are sufficiently close at a radius  $r_+ + \Delta r$ ,  $\Delta r > 0$  such that  $a_\Lambda(r_+ + \Delta r) < 1$ . So we consider the difference  $|a_\Lambda^2(r) - a^2(r)|$  at the radius  $r_+ + \Delta r$ :

$$(5.44) \quad |a_\Lambda^2(r_+ + \Delta r) - a^2(r_+ + \Delta r)| = |\Lambda| \frac{(r_+ + \Delta r)^2 + L_0}{3}.$$

Choosing  $|\Lambda|$  small one attains this difference to be smaller than  $a(r_+ + \Delta r) - 1$  which implies  $r_{+\Lambda} < r_+ + \Delta r < \infty$ .

Given this configuration (5.43) we construct a global solution of the Einstein-Vlasov system in the following manner. For  $r \in (r_{B\Lambda}, r_{+\Lambda}]$  we set  $f(x, v) \equiv 0$  and

$$(5.45) \quad \mu_\Lambda(r) = \frac{1}{2} \ln \left( 1 - \frac{r^2\Lambda}{3} - \frac{2M_0}{r} \right).$$

For  $r \geq r_{+\Lambda}$  we set  $f(x, v) = \Phi(E, L)$ . Since also  $\Phi(E, L) = 0$  on the interval  $(r_{-\Lambda}, r_{+\Lambda})$  the distribution function  $f$  is continuous and the metric coefficient  $\mu_\Lambda$  is given by the ODE (5.24) with  $\Lambda < 0$  for all  $r \in (r_{-\Lambda}, \infty)$ . The initial value  $\lambda_0$  of  $\lambda_\Lambda$  is determined by the continuity criterion

$$(5.46) \quad \frac{r_{+\Lambda}}{2} \left( 1 - e^{-2\lambda_0} \right) = M_0.$$

The last step of the proof is to assure for the existence of a solution with the desired properties of the Einstein-Vlasov system on  $[r_{+\Lambda}, \infty)$  with initial values  $\lambda_0$  given by (5.46) and  $\mu_0 = \mu_\Lambda(r_{+\Lambda})$ , given by equation (5.45). But this is already implied by the Theorems 4.2 and 4.3.  $\square$

## 6. SOLUTIONS ON $\mathbb{R} \times S^3$ AND $\mathbb{R} \times S^2 \times \mathbb{R}$

In Sections 3.4 and 5.1 we constructed spherically symmetric, static solutions of the Einstein-Vlasov system with small positive cosmological constant  $\Lambda$ . For small radii the  $\Lambda$ -term plays only a minor role. This was crucial for the method of proof. However, the global structure of the constructed spacetime is substantially different when  $\Lambda > 0$  and shows interesting properties. In particular, it allows for solutions with different global topologies.

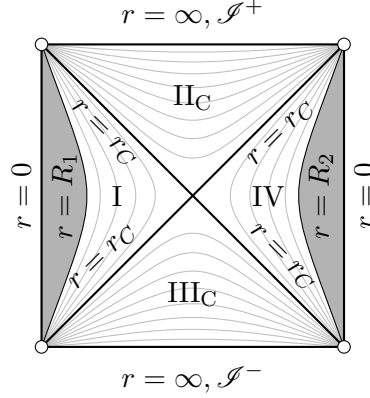
The following theorem gives a class of new solutions to the non-vacuum field equations with non-trivial global topology. These solutions are constructed from pieces consisting of solutions constructed in Theorems 3.7 and 5.5.

**Theorem 6.1.** *Let  $\Lambda > 0$  be sufficiently small and let  $\mathcal{M}_1 = \mathbb{R} \times S^3$  and  $\mathcal{M}_2 = \mathbb{R} \times S^2 \times \mathbb{R}$ . The following types of static metrics solving the Einstein-Vlasov system exist on these topologies.*

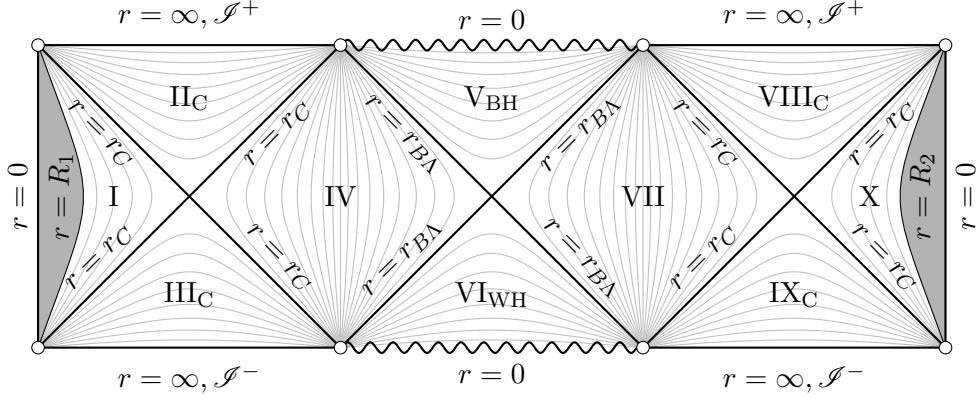
- (i) *There is a class of static metrics on  $\mathcal{M}_1$ , which is characterized in Figure 2. In regions I and IV a metric in this class coincides with two a priori different solutions of the type constructed in Theorem 3.7 with identical total mass, but possibly different matter distributions and radii of the support of the matter quantities  $R_1$  and  $R_2$  and regular centers. The metric in regions II and III is vacuum.*
- (ii) *There is a class of static metrics on  $\mathcal{M}_1$ , which is characterized in Figure 3. A metric in this class consists of two regular centers with finitely extended matter distribution around each of the centers of equal mass but possible different matter distributions and radii  $R_1, R_2$  of the type constructed in Theorem 3.7. These two regions are connected by a chain of black holes of identical masses (the diagram shows the minimal configuration with one black hole).*
- (iii) *There is a class of metrics on  $\mathcal{M}_2$ , which is characterized in Figure 4. The space-time consists of an infinite sequence of black holes, each surrounded by matter shells of possibly different radii and positions. In regions IV, VII, X and XIII these solutions coincide with those constructed in Theorem 5.5. The necessary conditions on the masses are  $M_\varrho^{A_1} = M_\varrho^{A_2}$ ,  $M_\varrho^{B_1} = M_\varrho^{B_2}$  and  $M_0^A + M_\varrho^{A_2} = M_\varrho^{B_1} + M_0^B$ , where  $M_0^i$ ,  $i = A, B$ , denote the mass parameter of the black holes and  $M_\varrho^{ij}$ ,  $i = A, B$ ,  $j = 1, 2$  denote the quasilocal mass of the matter shells defined in equation (6.11).*

### Remark 6.2.

- (a) *The black hole masses in the third class of solutions in the previous theorem can be pairwise different. Only the total mass of black hole and matter shell have to agree pairwise, cf. condition in (iii) above.*
- (b) *Combinations of the classes (ii) and (iii) yield similar metrics on  $\mathcal{M}_3 = \mathbb{R} \times \mathbb{R}^3$  with a regular center followed by an infinite sequence of black holes.*
- (c) *The second class of solutions could also be generalized by adding matter shells around the black holes. The mass parameters then have to be adjusted.*
- (d) *When crossing the cosmological horizon or the event horizon of a black or white hole the Killing vector  $\partial_t$  changes from being timelike to spacelike. This means that the maximal extended spacetime contains both static and dynamic regions that are alternating. This holds for all constructed classes.*



**Figure 2** – Penrose diagram of the maximal  $C^2$ -extension of a metric constructed as spherically symmetric solution of the Einstein-Vlasov system. Region I corresponds to the region  $0 < r < r_C$ . The metric is extended in an analogue way to the standard extension of the deSitter metric. The gray lines are surfaces of constant  $r$ .

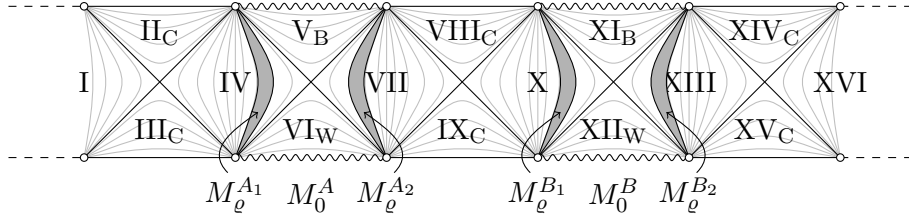


**Figure 3** – Penrose diagram of the maximal  $C^2$ -extension of a metric constructed as spherically symmetric solution of the Einstein-Vlasov system. Region I corresponds to the region  $0 < r < r_C$ . In this region matter (represented by the shaded area) is present and the metric is regular. This metric is extended with the Schwarzschild-deSitter metric that leads to a periodic solution. The periodic course stops when a matter region appears again preventing the metric from being singular at  $r = 0$ . The gray lines are surfaces of constant  $r$ .

*Proof.* We outline now the construction of the spacetimes given in the previous theorem. For the first two classes of spacetimes we consider solutions of the Einstein-Vlasov system with a regular center. Let  $(\mu_\Lambda, \lambda_\Lambda, f_\Lambda)$  be a static solution of the spherically symmetric Einstein-Vlasov system with positive cosmological constant  $\Lambda$  defined for  $r \in [0, r_C)$  such that the support of the matter quantities is bounded by a radius  $0 < R_{0\Lambda} < r_C$ . The radius  $r_C$  denotes the cosmological horizon. On  $[R_{0\Lambda}, r_C)$  there is vacuum on hand and the metric is given by the Schwarzschild-deSitter metric (6.4) with the ADM mass  $M$  as mass parameter. The ADM mass  $M$  is then given by

$$(6.1) \quad M = 4\pi \int_0^{R_{0\Lambda}} s^2 \varrho_\Lambda(s) ds.$$

If  $9M^2\Lambda < 1$ , the polynomial  $r^3 - \frac{3}{\Lambda}r + \frac{6M}{\Lambda}$  has one negative zero and two positive ones. The largest zero of this polynomial is defined to be the cosmological horizon  $r_C$ . Moreover,



**Figure 4** – Penrose diagram of the maximal  $C^2$ -extension of a metric constructed as spherically symmetric solution of the Einstein-Vlasov system. The solution coincides with the Schwarzschild-deSitter spacetime in the vacuum regions and the black holes are surrounded by shells of Vlasov matter (gray shaded domains). Notably the black holes do not necessarily have the same mass. The grey lines are surfaces of constant  $r$ .

$r_n$  is the negative zero, and  $r_{B\Lambda}$  the smaller positive one. In terms of the ADM mass  $M$  and the cosmological constant  $\Lambda$  these zeros can be calculated explicitly. Note that the Buchdahl inequality for solutions with  $\Lambda \neq 0$  [6] implies  $r_{B\Lambda} < R_{0\Lambda}$ .

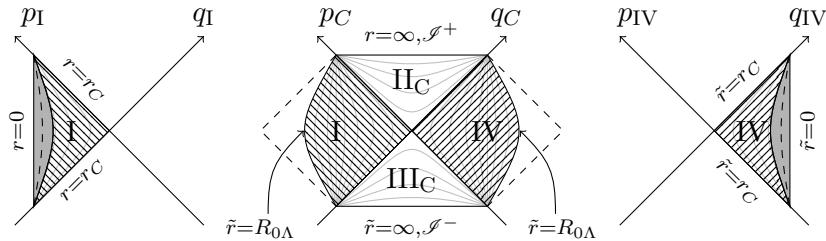
Case (i): Consider Figure 2. This spacetime can be obtained in an analogue way to the standard procedure to compactify the deSitter space as described for example in [17]. In the following, this procedure is carried out in detail. The metric is given as a non-vacuum solution of the Einstein-Vlasov system for  $r \in [0, r_C)$ , corresponding to region I in Figure 2, as discussed in Theorem 3.7. In this region we have

$$(6.2) \quad ds^2 = -e^{2\mu(r)} dt^2 + e^{2\lambda(r)} dr^2 + r^2 d\vartheta^2 + r^2 \sin^2(\vartheta) d\varphi^2.$$

In the first step we introduce coordinates  $U_I, V_I$  that transform the region  $\mathbb{R} \times [0, r_C) \times S^2$  into the left triangle (region I) in Figure 2. The coordinates usually used to compactify the vacuum deSitter metric as for example described in [17] will do. They are given by

$$(6.3) \quad U_I = \sqrt{\frac{r_C - r}{r_C + r}} e^{-\frac{t}{r_C}}, \quad V_I = -\sqrt{\frac{r_C - r}{r_C + r}} e^{\frac{t}{r_C}}$$

and can be compactified via the transformations  $p_I = \arctan(U_I)$ ,  $q_I = \arctan(V_I)$ . The left part of Figure 5 shows the transformed region  $\mathbb{R} \times [0, r_C)$  in the  $p_I, q_I$ -coordinates.



**Figure 5** – Construction of the spacetime shown in Figure 2. We use three coordinate charts to compactify the spacetime. Regions that are shaded in the same orientation are covered by two of the coordinate charts simultaneously, thus there coordinates can be changed. The gray areas are matter regions and the dashed lines correspond to  $r = r_{B\Lambda}$ . We distinguish between  $r$  and  $\tilde{r}$  to emphasize that there are different spacetime regions that cannot be covered by a single chart  $(t, r, \vartheta, \varphi)$ . All coordinates  $p$  and  $q$  take values in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

The support of the matter (i.e. the matter distribution  $f$ ) ends at a radius  $R_{0\Lambda}$ . For  $r \geq R_{0\Lambda}$  the metric is merely given by the Schwarzschild-deSitter metric

$$(6.4) \quad ds^2 = - \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M}{r} \right) dt^2 + \frac{dr^2}{1 - \frac{r^2 \Lambda}{3} - \frac{2M}{r}} + r^2 d\Omega^2, \quad R_{0\Lambda} \leq r < r_C.$$

At  $r = r_C$  there is a coordinate singularity of the metric that we want to pass. For this purpose we express the metric in other coordinates that do not have a singularity at  $r = r_C$  being defined on the region where  $r \in [R_{0\Lambda}, r_c)$  (region I in the middle part of Figure 5). These coordinates are given by

$$(6.5) \quad U_C = \sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^\gamma}} e^{-\frac{t}{2\delta_C}} > 0, \quad V_C = -\sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^\gamma}} e^{\frac{t}{2\delta_C}} < 0,$$

where  $\delta_C = \frac{r_C}{\Lambda r_C^2 - 1} > 0$  and  $\gamma = \frac{r_{B\Lambda}}{(1 - \Lambda r_{B\Lambda}^2)\delta_C}$ ,  $0 < \gamma < 1^3$ . They are used in the standard compactification procedure of the Schwarzschild-deSitter metric. For details, see [12] or [14]. In the new coordinates the line element of the Schwarzschild-deSitter metric (6.4) reads

$$(6.6) \quad ds^2 = -\frac{4\Lambda\delta_C^2}{3r}(r - r_n)^{2-\gamma}(r - r_{B\Lambda})^{1+\gamma}dU_CdV_C + r^2d\vartheta^2 + r^2\sin^2(\vartheta)d\varphi^2, \quad r \geq R_{0\Lambda}.$$

Note that here  $r$  is seen as a function of  $U_C$  and  $V_C$ . The coordinates only take values in  $\{(u, v) \in \mathbb{R}^2 \mid u > 0, v < 0\}$ . We extend them to  $\mathbb{R}^2$ . This extension gets beyond  $r_C$ . Again, the spacetime region covered by the coordinates  $U_C$  and  $V_C$  can be compactified using the transformation  $p_C = \arctan(U_C)$ ,  $q_C = \arctan(V_C)$ . The middle part of Figure 5 shows the region covered by  $U_C$  and  $V_C$ , each taking values in  $\mathbb{R}$ , in the  $p_C, q_C$ -coordinates. The line element (6.6) can be extended to the whole area covered by  $U_C$  and  $V_C$  in an analytic way. In the region where  $r \in [R_{0\Lambda}, r_C)$  the coordinate charts (6.3) and (6.5) overlap and one can change coordinates (the shaded areas in the left and middle part of Figure 5). The transformation law is given by

$$(6.7) \quad \begin{aligned} U_C(U_I) &= \sqrt{\frac{(r_C + r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^\gamma}} e^{\frac{3-2\Lambda r_C^2}{r_C}} U_I, \\ V_C(V_I) &= \sqrt{\frac{(r_C + r)(r - r_n)^{\gamma-1}}{(r - r_{B\Lambda})^\gamma}} e^{-\frac{3-2\Lambda r_C^2}{r_C}} V_I. \end{aligned}$$

Region IV in Figure 2 corresponds to a second universe that also can be equipped with Schwarzschild coordinates  $(\tilde{t}, \tilde{r})$ . We distinguish between  $r$  and  $\tilde{r}$  to emphasize that the charts  $(t, r)$  and  $(\tilde{t}, \tilde{r})$  cover different regions of the spacetime. Geometrically these regions look equal. This will be different for the second class of spacetimes (ii). In the region  $\tilde{r} \in [R_{0\Lambda}, r_C)$  (region IV in the middle part of Figure 5), in terms of the  $\tilde{t}, \tilde{r}$ -coordinates  $U_C$  and  $V_C$  are given by

$$(6.8) \quad U_C = -\sqrt{\frac{(r_C - \tilde{r})(\tilde{r} - r_-)^{\gamma-1}}{(\tilde{r} - r_{B\Lambda})^\gamma}} e^{-\frac{\tilde{t}}{2\delta_C}} < 0, \quad V_C = \sqrt{\frac{(r_C - \tilde{r})(\tilde{r} - r_-)^{\gamma-1}}{(\tilde{r} - r_{B\Lambda})^\gamma}} e^{\frac{\tilde{t}}{2\delta_C}} > 0.$$

To get a compactification of the whole region IV, including  $\tilde{r} < r_B$ , we introduce coordinates similar to (6.3), namely

$$(6.9) \quad U_{IV} = -\sqrt{\frac{r_C - \tilde{r}}{r_C + \tilde{r}}} e^{-\frac{\tilde{t}}{r_C}}, \quad V_{IV} = \sqrt{\frac{r_C - \tilde{r}}{r_C + \tilde{r}}} e^{\frac{\tilde{t}}{r_C}}$$

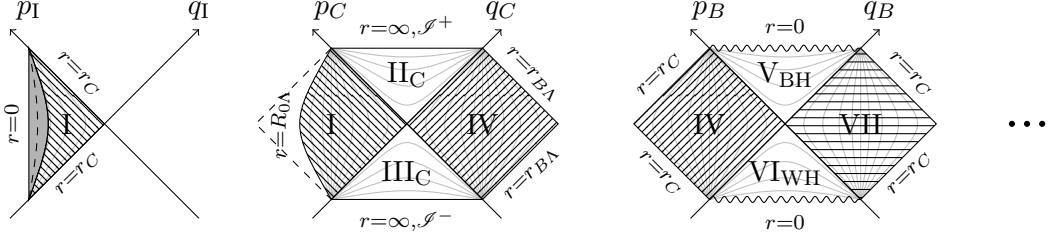
covering the region characterized by  $\tilde{r} \in [0, r_C)$ . This region can again be compactified via  $p = \arctan(U)$ ,  $q = \arctan(V)$ . This yields the right part of Figure 5. For  $\tilde{r} \in [R_{0\Lambda}, r_C)$

<sup>3</sup>The signs of these expressions can be checked with the equality  $1 - \frac{r_C^2 \Lambda}{3} - \frac{2M}{r_C} = 0$



the coordinates can be changed using an analogue law as (6.8). On the spacetime region represented by the middle part of Figure 5 the line element can be expressed by (6.6). Since both, in region I and IV the metric can be brought into the form (6.2) via coordinate transformations also the energy densities are identical in these regions. This of course implies that in both regions the mass parameter is equal.

Case (ii): Now we come to the spacetimes characterized by Figure 3. For the construction of a  $C^2$ -extension of the metric (6.2) at least five coordinate charts are necessary. Figure 6 illustrates this construction. Again we begin with the region  $r \in [0, r_C)$  where the metric



**Figure 6** – Construction of the spacetime shown in Figure 3. On regions that are shaded in equal directions two coordinates are defined and one can change between them. All coordinates  $p, q$  take values in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

is given by (6.2). In the same way as described above one expresses the line element in other coordinates  $p_C, q_C$  that avoid the singularity at  $r = r_C$  and cover the region  $r_{0\Lambda} < r < r_C$ . The line element as given by (6.6) can be analytically<sup>4</sup> extended onto the regions I – IV in Figure 6. From now on the procedure differs from the one above. Regions I and IV are not supposed to be geometrically identical but region IV shall be a vacuum region thus the metric will be given by Schwarzschild-deSitter everywhere. Certainly, the line element (6.6) of the Schwarzschild-deSitter metric being given in terms of the coordinates  $U_C, V_C$  now shows a singularity at  $r = r_{B\Lambda}$ <sup>5</sup>. This coordinate singularity can be overcome by virtue of the coordinates

$$(6.10) \quad U_B = \sqrt{\frac{(r - r_{B\Lambda})(r - r_n)^{\beta-1}}{(r_C - r)^\beta}} e^{\frac{t}{2\delta_B}}, \quad V_B = -\sqrt{\frac{(r - r_{B\Lambda})(r - r_n)^{\beta-1}}{(r_C - r)^\beta}} e^{-\frac{t}{2\delta_B}},$$

where  $\delta_B = \frac{r_{B\Lambda}}{1 - \Lambda r_{B\Lambda}^2} > 0$  and  $\beta = \frac{r_C}{(\Lambda r_C^2 - 1)\delta_B} > 1$ . The coordinates are defined on the middle part of Figure 6. This is part of the standard compactification procedure of the Schwarzschild-deSitter metric, cf. [12] or [14]. Alternating the coordinate charts  $(U_C, V_C)$  and  $(U_B, V_B)$  this procedure can be continued an arbitrary amount of times extending the spacetime to additional black hole and cosmological regions. This periodic extension stops if for  $r < r_C$  the metric is not given by a vacuum solution of the Einstein equations but again by the solution (6.2) of the Einstein-Vlasov system. There is no coordinate singularity at  $r = r_{B\Lambda}$  and also a regular center at  $r = 0$ . So a regular expression of the line element by the coordinates (6.3) is possible again, leading to region X in Figure 3. This region now is geometrically identical to region I in Figure 3 (and also in Figure 6). In the extension procedure just described the expressions for the coordinates (6.8) and (6.10) used to pass the coordinate singularities at  $r = r_{B\Lambda}$  and  $r = r_C$  in the vacuum regions of the spacetime  $\mathcal{M}_1$  depend on  $\Lambda$  and  $M$ . So the identification of corresponding regions in the different coordinate charts, e.g. I or IV in Figure 6, is only possible if the

<sup>4</sup>In matter regions the regularity of the metric is  $C^2$  as provided by Theorem 3.7, in vacuum regions the metric is analytic.

<sup>5</sup>By abuse of notation we use  $r$  for the radius coordinate in every region of the spacetime  $\mathcal{M}_1$ .

parameters  $\Lambda$  and  $M$  are equal in all regions of  $\mathcal{M}_1$ . In terms of the notation of Figure 3 this implies  $M_1 = M_2$ .

Case (iii): A maximal extension of a solution to the Einstein-Vlasov system on the manifold  $\mathcal{M}_2$  as characterized by Figure 4, i.e. spacetimes in class (iii), can be obtained in a similar way. Starting point is the region  $r_{B\Lambda} < r < r_C$ . On this interval the existence of a unique solution to a given ansatz for  $f$  is established by Theorem 5.5. The solution on hand can be understood as a Schwarzschild-deSitter spacetime with an immersed shell of Vlasov matter supported on an interval  $(r_{+\Lambda}, R_{0\Lambda})$ . Two mass quantities are important. On the one hand one has the mass parameter  $M_0$  of the black hole at the center. On the other hand  $M$  that is defined to be

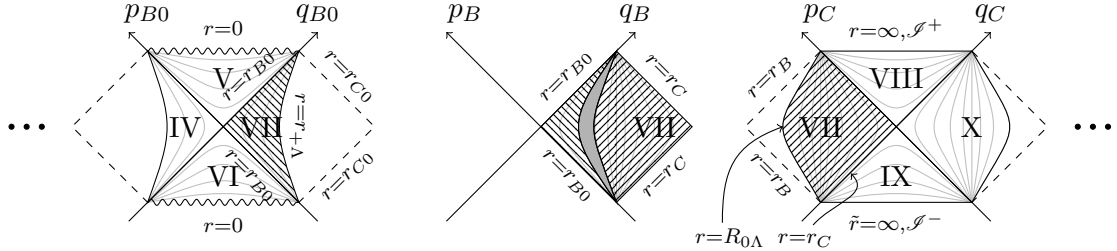
$$(6.11) \quad M = M_0 + M_\varrho, \quad M_\varrho = 4\pi \int_{r_{+\Lambda}}^{R_{0\Lambda}} s^2 \varrho_\Lambda(s) ds.$$

This quantity represents the sum of the mass of the black hole and the shell of Vlasov matter. As constructed in Theorem 5.5, for  $r_{B\Lambda} < r \leq r_{+\Lambda}$  the metric is given by a shifted Schwarzschild-deSitter metric

$$(6.12) \quad ds^2 = -C \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M_0}{r} \right) dt^2 + \frac{dr^2}{C \left( 1 - \frac{r^2 \Lambda}{3} - \frac{2M_0}{r} \right)} + r^2 d\Omega^2, \quad r_{B\Lambda} < r \leq r_{+\Lambda}$$

with the mass  $M_0$  of the black hole as mass parameter and the shift  $C > 0$ . For  $R_{0\Lambda} \leq r < r_C$  the metric is given by the Schwarzschild-deSitter metric (6.4) with mass parameter  $M$ .

The two critical horizons,  $r_{B\Lambda}$  and  $r_C$  can be given explicitly as zeros of the expression  $1 - \frac{r^2 \Lambda}{3} - \frac{2m(r)}{r}$ . But it is important to note that the mass parameter  $m(r)$  does not stay constant throughout the whole interval  $(r_{B\Lambda}, R_{0\Lambda})$ . The black hole horizon  $r_{B\Lambda}$  is characterized by  $M_0$  and the cosmological horizon  $r_C$  by  $M$ . This has to be kept in mind when choosing coordinates to construct an extension of the metric on  $\mathcal{M}_2$  as illustrated in Figure 7. We distinguish between the zeros of  $1 - \frac{r^2 \Lambda}{3} - \frac{2m(r)}{r}$  when  $m(r) \equiv M_0$  and



**Figure 7** – Construction of the spacetime shown in Figure 4. The middle part shows a Schwarzschild-deSitter spacetime with an immersed matter shell for  $r_{B\Lambda} = r_{B0} < r < r_C$ . The left and the right part show the adjacent vacuum region containing several coordinate singularities. On regions that are shaded in equal directions two coordinates are defined and one can change between them. All coordinates  $p, q$  take values in  $[-\frac{\pi}{2}, \frac{\pi}{2}]$ .

$m(r) \equiv M$  and call them  $r_{B0}, r_{C0}$  or  $r_B, r_C$ , respectively. Note that  $r_{B0} = r_{B\Lambda}$ . Consider the metric on the region  $r_{B0} < r < r_C$  being part of region VII in Figure 4 or the middle part of Figure 7. The metric shall be extended to the left (regions IV,  $V_B$ ,  $VI_W$ ) and to the right (regions  $VIII_C$ ,  $IX_C$ , X) as a vacuum solution until the next matter shell appears. So the coordinate transformations have to be chosen with respect to the radii  $r_B$  and  $r_C$  belonging to the current mass parameter in the respective spacetime region. Three coordinate charts are needed to extend the metric beyond the black hole and the

cosmological horizon. First we compactify the region  $r_{B\Lambda} = r_{B0} < r < r_C$  using the coordinates

$$(6.13) \quad U_B = \sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta-1}}{(r_C - r)^\beta}} e^{\frac{t}{2\delta_{B0}}}, \quad V_B = -\sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta-1}}{(r_C - r)^\beta}} e^{-\frac{t}{2\delta_{B0}}}.$$

where  $\delta_{B0} = \frac{r_{B0}}{1 - \Lambda r_{B0}^2} > 0$  and  $\beta = \frac{r_C}{(\Lambda r_C^2 - 1)\delta_{B0}} > 1$ . These coordinates give rise to  $p_B = \arctan(U_B)$  and  $q_B = \arctan(V_B)$ . This region is depicted in the middle part of Figure 7. The spacetimes characterized by Figure 4 show two types of connected vacuum regions. The first type is characterized by  $r \leq r_{+\Lambda}$  (inside the matter shell) and the second one by  $r \geq R_{0\Lambda}$  (beyond the matter shell). To extend the metric to the region inside the matter shell (and the black hole) one uses the coordinates

$$(6.14) \quad U_{B0} = \sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta_0-1}}{(r_{C0} - r)_0^\beta}} e^{\frac{t}{2\delta_{B0}}}, \quad V_{B0} = -\sqrt{\frac{(r - r_{B0})(r - r_n)^{\beta_0-1}}{(r_{C0} - r)_0^\beta}} e^{-\frac{t}{2\delta_{B0}}},$$

where  $\delta_{B0} = \frac{r_{B0}}{1 - \Lambda r_{B0}^2} > 0$  and  $\beta_0 = \frac{r_{C0}}{(\Lambda r_{C0}^2 - 1)\delta_{B0}} > 1$ , and the corresponding compactification  $p_{B0} = \arctan(U_{B0})$ ,  $q_{B0} = \arctan(V_{B0})$ . These coordinates are valid for  $0 < r < r_{+\Lambda}$ . The black hole horizon can be crossed using the usual arguments of the extension of the Schwarzschild-deSitter metric as for example done in [17, 12, 14]. This is illustrated in the left part of Figure 7. The region beyond the matter shell (and the cosmological horizon) can be reached via the coordinates

$$(6.15) \quad U_C = -\sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^\gamma}} e^{-\frac{t}{2\delta_C}}, \quad V_C = \sqrt{\frac{(r_C - r)(r - r_n)^{\gamma-1}}{(r - r_B)^\gamma}} e^{-\frac{t}{2\delta_C}},$$

with  $\delta_C = \frac{r_C}{\Lambda r_C^2 - 1} > 0$  and  $\gamma = \frac{r_B}{(1 - \Lambda r_B^2)\delta_C}$ ,  $0 < \gamma < 1$ . These coordinates extend the metric to the area  $R_{0\Lambda} < r < \infty$ , shown in the right part of Figure 7.

On the connected vacuum regions the metric is given by only one expression even though vacuum extends onto several regions of  $\mathcal{M}_2$ , e.g. regions VII, VIII<sub>C</sub>, IX<sub>C</sub> and X. This implies that the coordinates  $U_{B0}$ ,  $V_{B0}$  or  $U_C$ ,  $V_C$  have to be given by the same expressions (6.14) or (6.15), respectively (modulo sign, cf. [17, 12, 14]) which in turn implies that the mass parameter has to stay the same on these connected vacuum regions. For the vacuum region with  $r \geq R_{0\Lambda}$  this implies  $M_0^A + M_\varrho^{A_2} = M_0^B + M_\varrho^{B_1}$  (notation of Figure 4). On the region characterized by  $r \leq r_{+\Lambda}$  this is always granted because the mass is entirely given by the black hole mass  $M_0$ . Finally the shift constants  $C > 0$  of the vacuum metric have to coincide in this region (IV and VII in Figure 4). They are determined by the matter shells surrounding the black hole and are equal in particular if these shells have the same shape which implies  $M_\varrho^{A_1} = M_\varrho^{A_2}$ .  $\square$

#### APPENDIX A. PROOF THAT T ACTS AS A CONTRACTION

In order to show that the operator  $T$ , defined in (3.2) acts as a contraction on the set  $M$ , defined in (3.3), one has to check

- (a)  $u \equiv y_0 \in M$ ,
- (b)  $u \in M \Rightarrow Tu \in M$ , and
- (c)  $\exists a \in (0, 1) \forall u, v \in M : \|Tu - Tv\|_{\infty, \delta} \leq a\|u - v\|_{\infty, \delta}$ , where  $\|\cdot\|_{\infty, \delta} = \sup_{r \in [0, \delta]}(\cdot)$ .

(i): Consider  $u \equiv y_0$ . Only the second critical condition

$$(A.1) \quad \frac{r^2 \Lambda}{3} + \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, u(s)) ds \leq c$$

is relevant. We calculate

$$\frac{r^2 \Lambda}{3} + \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, u(s)) ds \leq \frac{r^2 \Lambda}{3} + \frac{\kappa r^2}{3} G_\phi(\delta, y_0) \leq \frac{\Lambda + \kappa G_\phi(r, y_0)}{3} \delta^2 \leq c$$

for  $\delta$  small enough.

(ii): We have to guarantee that  $y_0 - 1 \leq (Tu)(r) \leq y_0 + 1$  and

$$\frac{r^2 \Lambda}{3} + \frac{\kappa}{r} \int_0^r s^2 G_\phi(s, Tu(s)) ds \leq c.$$

By choosing  $\delta$  sufficiently small, one can achieve the domain of integration in  $T$  to become arbitrarily small and these properties follow.

(iii): We calculate

$$\begin{aligned} & \|Tu - Tv\|_{\infty, \delta} \\ &= \left\| \int_0^r \left[ \frac{\kappa/2}{1 - \frac{s^2 \Lambda}{3} - \frac{\kappa}{s} \int_0^s \sigma^2 G_\phi(\sigma, u(\sigma)) d\sigma} \left( s(H_\phi(s, u(s)) - H_\phi(s, v(s))) \right. \right. \right. \\ & \quad \left. \left. \left. + \frac{1}{s^2} \int_0^s \sigma^2 (G_\phi(\sigma, u(\sigma)) - G_\phi(\sigma, v(\sigma))) d\sigma \right) \right. \right. \\ & \quad \left. \left. + \left( sH_\phi(s, v(s)) - \frac{2s\Lambda}{3\kappa} + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, v(\sigma)) d\sigma \right) \right] ds \right\|_{\infty, \delta}. \end{aligned}$$

Since  $G_\phi(r, u)$ ,  $H_\phi(r, u)$ ,  $\partial_u G_\phi(r, u)$ , and  $\partial_u H_\phi(r, u)$  are strictly in  $u$  increasing functions, we have

$$\begin{aligned} \sup_{u \in [y_0 - 1, y_0 + 1]} H_\phi(r, u) &= H_\phi(r, y_0 + 1) =: H_{\text{sup}}(r), \\ \sup_{u \in [y_0 - 1, y_0 + 1]} G_\phi(r, u) &= G_\phi(r, y_0 + 1) =: G_{\text{sup}}(r), \\ \sup_{u \in [y_0 - 1, y_0 + 1]} |\partial_u H_\phi(r, u)| &= |\partial_u H_\phi(r, y_0 + 1)| =: G'_{\text{sup}}(r), \\ \sup_{u \in [y_0 - 1, y_0 + 1]} |\partial_u G_\phi(r, u)| &= |\partial_u G_\phi(r, y_0 + 1)| =: H'_{\text{sup}}(r). \end{aligned}$$

We can estimate the first summand in the following way:

$$\begin{aligned} & \int_0^r \frac{\kappa/2}{1 - \frac{s^2 \Lambda}{3} - \frac{\kappa}{s} \int_0^s \sigma^2 G_\phi(\sigma, u(\sigma)) d\sigma} \\ & \quad \times \left( s(H_\phi(s, u(s)) - H_\phi(s, v(s))) + \frac{1}{s^2} \int_0^s \sigma^2 (G_\phi(\sigma, u(\sigma)) - G_\phi(\sigma, v(\sigma))) d\sigma \right) ds \\ & \leq \frac{\kappa}{2(1-c)} \frac{\delta^2}{2} \left( H'_{\text{sup}}(\delta) + \frac{1}{3} G'_{\text{sup}}(\delta) \right) \|u - v\|_{\infty, \delta}. \end{aligned}$$

Next, we consider the second summand:

$$\begin{aligned} & \int_0^r \left( sH_\phi(s, v(s)) - \frac{2s\Lambda}{3\kappa} + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, v(\sigma)) d\sigma \right) \\ & \quad \times \left( \frac{\kappa/2}{1 - \frac{s^2 \Lambda}{3} - \frac{\kappa}{s} \int_0^s \sigma^2 G_\phi(\sigma, u(\sigma)) d\sigma} - \frac{\kappa/2}{1 - \frac{s^2 \Lambda}{3} - \frac{\kappa}{s} \int_0^s \sigma^2 G_\phi(\sigma, v(\sigma)) d\sigma} \right) ds \\ & \leq \int_0^r s \left( H_{\text{sup}}(r) - \frac{2\Lambda}{3\kappa} + \frac{1}{3} G_{\text{sup}}(r) \right) \frac{\kappa^2 s^2}{6(1-2c+c^2)} ds G'_{\text{sup}}(r) \|u - v\|_{\infty, \delta} \\ & \leq \frac{\kappa^2 \delta^4}{24(1-2c+c^2)} \left( H_{\text{sup}}(\delta) - \frac{2\Lambda}{3\kappa} + \frac{1}{3} G_{\text{sup}}(\delta) \right) G'_{\text{sup}}(\delta) \|u - v\|_{\infty, \delta}. \end{aligned}$$

So we get in total

$$\begin{aligned} \|Tu - Tv\|_{\infty, \delta} &\leq \left( \frac{\kappa}{4(1-c)} \left( H'_{\text{sup}}(\delta) + \frac{1}{3} G'_{\text{sup}}(\delta) \right) \delta^2 \right. \\ &\quad \left. + \frac{\kappa^2}{24(1-2c+c^2)} \left( H_{\text{sup}}(\delta) - \frac{2\Lambda}{3\kappa} + \frac{1}{3} G_{\text{sup}}(\delta) \right) G'_{\text{sup}}(\delta) \delta^4 \right) \|u - v\|_{\infty, \delta}. \end{aligned}$$

If one actually wants to calculate  $\delta$  one can make use of the estimate

$$\begin{aligned} G_\phi(r, u) &= c_\ell r^{2\ell} \int_{\sqrt{1+L_0/r^2}}^{\infty} \phi(1 - \varepsilon e^{-y}) \varepsilon^2 \left( \varepsilon^2 - \left( 1 + \frac{L_0}{r^2} \right) \right)^{\ell + \frac{1}{2}} d\varepsilon \\ (A.2) \quad &\leq c_\ell r^{2\ell} \int_1^{\infty} \phi(1 - \varepsilon e^{-y}) \varepsilon^2 (\varepsilon^2 - 1)^{\ell + \frac{1}{2}} d\varepsilon \end{aligned}$$

and the analogue one for  $H_\phi$  to obtain a polynomial in  $\delta$ .

#### APPENDIX B. ESTIMATE OF $|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|$

The following calculation is valid for  $r \in [0, \tilde{r}^*]$  where we can take for granted  $1 - \frac{2m(r)}{r} \geq \frac{1}{9}$  (Buchdahl inequality, cf. [4]),  $1 - \frac{r^2\Lambda}{3} - \frac{2m_\Lambda(r)}{r} \geq \frac{1}{18}$  and  $|y_\Lambda(r) - y(r)| \leq |y(R_0 + \Delta R)|$ ,  $\Delta R > 0$  where  $R_0$  is defined to be the (first) zero of the background solution  $y$ . Since

$$\begin{aligned} (B.1) \quad &|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)| \\ &\leq \left( \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u G_\phi(r, u)| + \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u H_\phi(r, u)| \right) |y_\Lambda(r) - y(r)| \end{aligned}$$

we calculate

$$\begin{aligned} |y_\Lambda(r) - y(r)| &\leq \int_0^r |y'(s) - y'_\Lambda(s)| ds \\ &\leq \int_0^r \left[ \underbrace{\frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}}}_{\leq 72\pi} \right. \\ &\quad \times \left( \left| -\frac{s\Lambda}{12\pi} \right| + s |H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| \right. \\ &\quad \left. \left. + \underbrace{\frac{1}{s^2} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| d\sigma}_{I_1} \right) \right. \\ &\quad \left. + \left( s H_\phi(s, y(s)) + \frac{1}{s^2} \int_0^s \sigma^2 G_\phi(\sigma, y(\sigma)) d\sigma \right) \underbrace{\left( \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}} - \frac{4\pi}{1 - \frac{2m(s)}{s}} \right)}_{I_2} \right] ds. \end{aligned}$$

We estimate  $I_1$  and  $I_2$  separately:

$$\begin{aligned} I_1 &= \int_0^r \frac{1}{s^2} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| d\sigma ds \\ &\leq \int_0^r \int_0^r |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| d\sigma ds \\ &\leq r \int_0^r |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| d\sigma, \end{aligned}$$

$$\begin{aligned}
I_2 &= \frac{4\pi}{1 - \frac{s^2\Lambda}{3} - \frac{2m_\Lambda(s)}{s}} - \frac{4\pi}{1 - \frac{2m(s)}{s}} \\
&\leq 4\pi \cdot 18 \cdot 9 \cdot \left( \frac{s^2\Lambda}{3} + \frac{8\pi}{s} \int_0^s \sigma^2 |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| d\sigma \right) \\
&\leq 648\pi \left( \frac{s^2\Lambda}{3} + 8\pi s \int_0^s |G_\phi(\sigma, y_\Lambda(\sigma)) - G_\phi(\sigma, y(\sigma))| d\sigma \right).
\end{aligned}$$

So using that  $y$  is decreasing we have

$$\begin{aligned}
&|y_\Lambda(r) - y(r)| \\
&\leq \Lambda \int_0^r \left( 6s + 216\pi s^3 \left( H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) ds \\
&\quad + 72\pi r \int_0^r |H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| ds \\
&\quad + \left( 72\pi r + 5184\pi^2 \frac{r^3}{3} \left( H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \int_0^r |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))| ds \\
&\leq \Lambda \left( 3r^2 + 54\pi r^4 \left( H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \\
&\quad + \left( 72\pi r + 1728\pi^2 r^3 \left( H_\phi(r, y_0) + \frac{1}{3} G_\phi(r, y_0) \right) \right) \\
&\quad \times \int_0^r (|H_\phi(s, y_\Lambda(s)) - H_\phi(s, y(s))| + |G_\phi(s, y_\Lambda(s)) - G_\phi(s, y(s))|) ds \\
&\leq \Lambda C_1(r) + C_2(r) \int_0^r (|p_\Lambda(s) - p(s)| + |\varrho_\Lambda(s) - \varrho(s)|) ds
\end{aligned}$$

The derivatives with respect to  $y$  of  $G_\phi(r, y)$  and  $H_\phi(r, y)$  are strictly increasing both in  $r$  and  $y$ . And since  $|y_\Lambda(r) - y(r)| \leq |y(R_0 + \Delta R)|$  we can write

$$\begin{aligned}
&\left( \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u G_\phi(r, u)| + \sup_{u \in [y_\Lambda(r), y(r)]} |\partial_u H_\phi(r, u)| \right) \\
&\leq |\partial_u G_\phi(\tilde{r}^*, u)|_{y_0 + |y(R_0 + \Delta R)|} + |\partial_u H_\phi(\tilde{r}^*, u)|_{y_0 + |y(R_0 + \Delta R)|} =: C_3.
\end{aligned}$$

So we have obtained that equation (B.1) is of the form

$$|p_\Lambda(s) - p(s)| + |\varrho_\Lambda(s) - \varrho(s)| \leq C_4(r)\Lambda + C_5(r) \int_0^r (|p_\Lambda(s) - p(s)| + |\varrho_\Lambda(s) - \varrho(s)|) ds$$

Note that  $C_4(r)$  is strictly increasing. Grönwall's inequality yields

$$(B.2) \quad (|\varrho_\Lambda(r) - \varrho(r)| + |p_\Lambda(r) - p(r)|) \leq C_4(r) e^{\int_0^r C_5(s) ds} = C_4(r) \Lambda e^{r C_5(r)} =: C_{gh}(r) \Lambda.$$

Note that  $C_{gh}(r)$  is increasing when  $r$  is increasing.

## REFERENCES

- [1] L. ANDERSSON, R. BEIG & B. SCHMIDT, *Static self-gravitating elastic bodies in Einstein gravity*, Commun. Pure Appl. Math. **61**, 988-1023 (2008)
- [2] H. ANDRÉASSON, *The Einstein-Vlasov System/Kinetic Theory*, Living Rev. Relativity **24**, (2011), 4
- [3] H. ANDRÉASSON, *On static shells and the Buchdahl inequality for the spherically symmetric Einstein-Vlasov system*, Comm. Math. Phys. **274**, 409-425 (2007).
- [4] H. ANDRÉASSON, *Sharp bounds on  $2m/r$  of general spherically symmetric static objects*, Jour. Diff. Eq., Vol. **245**, Issue 8, 2243-2266 (2008)
- [5] H. ANDRÉASSON, *Sharp Bounds in the Critical Stability Radius for Relativistic Charged Spheres*, Commun. Math. Phys. **288**, 715-730 (2009)
- [6] H. ANDRÉASSON & C.G. BÖHMER, *Bounds on  $M/R$  for static objects with a positive cosmological constant*, Classical Quantum Gravity **26**, 195007 (2009).

- [7] H. ANDRÉASSON & C.G. BÖHMER & A. MUSSA, *Bounds on  $M/R$  for charged objects with positive cosmological constant*, Classical Quantum Gravity **29**, 095012 (2012).
- [8] H. ANDRÉASSON & G. REIN, *On the steady states of the spherically symmetric Einstein-Vlasov system*, Class. Quantum Grav. **24**, 1809-1832 (2007)
- [9] H. ANDRÉASSON & M. KUNZE & G. REIN, *Existence of Axially Symmetric Static Solutions of the Einstein-Vlasov system*, Commun. Math. Phys. **308**, 23-47 (2011)
- [10] H. ANDRÉASSON & M. KUNZE & G. REIN, *Rotating, stationary, axially symmetric spacetimes with collisionless matter*, Commun. Math. Phys. **329**, 787-808 (2014)
- [11] J. BATT, W. FALTENBACHER, & E. HORST, *Stationary Spherically Symmetric Models in Stellar Dynamics*, Arch. Rational Mech. Anal. **93**, 159-183 (1986)
- [12] S. L. BAŻAŃSKI & V. FERRARI, *Analytic Extension of the Schwarzschild-de Sitter Metric*, Il Nuovo Cimento Vol. **91** B, No.1 (1986)
- [13] G.L. BUNTING & A.K.M. MASOOD-UL-ALAM, *Nonexistence of multiple black holes in asymptotically Euclidean static vacuum space-time*, General Relativity and Gravitation, **19**, 2, (1987), 147-154
- [14] C. STANCIULESCU, *Spherically Symmetric Solutions of the Vacuum Einstein Field Equations with Positive Cosmological Constant*, diploma thesis at the University of Vienna (1998)
- [15] M. DAFERMOS & A. RENDALL, *An extension principle for the Einstein-Vlasov system in spherical symmetry*, Ann. Henri Poincaré **6**, 1137-1155 (2005)
- [16] G. F. R. ELLIS, S. T. C. SIKLOS & J. WAINWRIGHT, *Dynamical Systems in Cosmology*, 1997, Cambridge University Press
- [17] G. W. GIBBONS, S. W. HAWKING, *Cosmological event horizons, thermodynamics, and particle creation*, 1977, Phys. Rev. D, **15**, 10, 2738-2751
- [18] S. W. HAWKING & H. S. REALL, *Charged and rotating AdS black holes and their CFT duals*, Phys. Rev. D **61** (2000), 024014, hep-th/9908109
- [19] P. KARAGEORGIS & J. STALKER, *Sharp bounds on  $2m/r$  for static spherical objects*, Class. Quantum Grav. **25** (2008) 195021
- [20] T. RAMMING & G. REIN, *Spherically symmetric equilibria for self-gravitating kinetic or fluid models in the non-relativistic case – A simple proof for finite extension*, SIAM J. Math. Anal. **45** (2), 900 - 914, 2013
- [21] G. REIN, A.D. RENDALL, *Compact support of spherically symmetric equilibria in non-relativistic and relativistic galactic dynamics*, Math. Proc. Camb. Phil. Soc. (2000), **128**, 363
- [22] G. REIN, *Static shells for the Vlasov-Poisson and Vlasov-Einstein system*, Indiana University Mathematics Journal **48**, 335-346 (1999)
- [23] G. REIN, *Static solutions of the spherically symmetric Vlasov-Einstein system*, Math. Proc. Camb. Phil. Soc. (1994), **115**, 559
- [24] G. REIN & A. D. RENDALL, *Smooth static solutions of the spherically symmetric Vlasov-Einstein system*, Annales de l'I.H.P., section A, tome 59, n° 4, 383-397 (1993)
- [25] G. REIN & A. D. RENDALL, *Global Existence of Solutions of the Spherically Symmetric Vlasov-Einstein System with Small Initial Data*, Comm. Math. Phys. **150**, 561-583 (1992)
- [26] H. RINGSTRÖM, *On the topology and future stability of the universe*, Oxford Mathematical Monographs, 2013
- [27] K. SCHWARZSCHILD, *Über das Gravitationsfeld eines Massepunktes nach der Einsteinschen Theorie*, Sitzungsber. d. Preuss. Akad. d. Wiss. **7** p. 189, 1916
- [28] G. WOLANSKY, *Static Solutions of the Vlasov-Einstein System*, Arch. Rational Mech. Anal. **156** (2001) 205-230

HÅKAN ANDRÉASSON, CHALMERS UNIVERSITY OF TECHNOLOGY AND UNIVERSITY OF GOTHENBURG

hand@chalmers.de

DAVID FAJMAN, UNIVERSITY OF VIENNA

David.Fajman@univie.ac.at

MAXIMILIAN THALLER, UNIVERSITY OF VIENNA

Maximilian.Thaller@arcor.de